INTERNATIONAL DESIGN CONFERENCE - DESIGN 2014
Dubrovnik - Croatia, May 19 - 22, 2014.

ESTIMATING UNCERTAINTY OF SCALED PRODUCTS USING SIMILARITY RELATIONS AND LAWS OF GROWTH

J. Lotz, H. Kloberdanz, T. Freund and K. Rath

Keywords: scaling, uncertainty, Monte-Carlo simulation, laws of growth

1. Introduction

This paper presents an initial attempt at scaling the uncertainty caused by deviations in product properties due to production processes. There are no methods or methodologies that describe how to handle uncertainty, especially when scaling a product. Having an understanding of how to handle uncertainty is necessary. Particularly because of cases where production technologies do not create equal quality when scaling the size of the product that has to be machined. Depending on the product's size, the tolerances (deviations from planned properties and therefore uncertainty) change and their relative size compared to the size of the product is not constant. This paper gives an overview of how products are scaled, how production-related uncertainty can be classified, and how its influence on product behaviour can be calculated and analysed. Calculation of the uncertain product properties of scaled products is performed using laws of growth and Monte Carlo simulations. A guide on how to analyse the results and advice for product designers working on scaling problems are given.

2. Fundamentals

This section gives an overview of the methods, classifications and example products used in this paper.

2.1 Scaling products using a law of growth

The scaling of products is a necessity for many designers due to customer needs, for example, waterpumps in a wide range of power levels are often needed. Scaling in many cases deals with geometrical scaling of one design which is identical for all type sizes. Scaling can also be at the power level, for example, using more powerful actuators, or scaling stiffness of a spring by using different materials, etc.
With that in mind, this paper gives an overview of one of the two scaling methods: the laws of growth according to Pahl et al. [2007]. The other common way to scale a product is to retain similarity by using similarity factors produced by a dimensional analysis [Gibbings 2011]. It is possible to predict a product's properties with ease if the physical laws that describe the problem's relevant effects are known and can be handled using analysis, or if enough empirical data are available to create an approximation. In these cases, laws of growth can be found. An example to demonstrate how a law of growth is built is an ideal beam (no predeformation, perfectly straight and homogenous) with a round cross-section. The product property that should be scaled is the critical force for buckling due to axial compression load. According to Euler [Gross et al. 2012] this critical load for size variant *I* (index *i*) is given as

$$F_{crit,i} = \frac{E_i \cdot I_i}{l_i^2} \tag{1}$$

with I_i as the second area moment of inertia, depending on the beam's diameter d and its length l. The factor of growth related to the basic draft (index "0") for any scaling of those parameters is given by

$$\frac{F_{crit,i}}{F_{crit,0}} = \frac{E_i \cdot I_i \cdot l_0^2}{E_0 \cdot I_0 \cdot l_i^2} \tag{2}$$

In this equation, the ratio between two product properties of the same kind is called the step factor [Pahl et al. 2007] and is written as φ_j where j represents the property. φ_d refers to the diameter of the beam, φ_E to its Young's modulus. Taking into account that I depends on d in the fourth power makes Equation (2)

$$\phi_{F_{crit}} = \phi_d^4 \cdot \phi_E^1 \cdot \phi_l^{-2} \tag{3}$$

which is now the law of growth for the critical load in common notation. If geometrical similarity is retained, the step factors for length and diameter are equal. Young's modulus may or may not be dependent on size. Using nano wires made from gold, Young's modulus was not dependent on the wire diameter, but on its microstructure [Roehling 2011]. Zinc oxide shows a large size dependency on micro and nano scales [Chen et al. 2006]. So scaling to micro and nano scales, a possible size dependency of Young's module has to be checked. At scales where Young's modulus stays nearly constant, Equation (3) gets reduced to

$$\phi_{F_{crit}} = \phi_l^2 \tag{4}$$

With this result, it is possible to calculate the necessary geometrical scaling factor to achieve a determined critical load. It is also possible to construct laws of growth for semi-similar problems. For example, if the length always has the same value, its step factor just has to be set to one.
Based upon this method a way to determine and quantify known uncertainty is then developed.

2.2 Classification of examined types of uncertainty

Product designers have to deal with uncertainty throughout the whole product life cycle. According to Hanselka et al., "Uncertainty occurs when process properties of a system can not be determined" [Hanselka et al. 2010]. Uncertainty can be classified by type, depending on the information available to the designer [Hanselka et al. 2010].
In this paper, two of the three types of uncertainty given in the cited paper are examined. One is stochastical uncertainty, which occurs mostly in the later stages of product development or within production processes and usage. Stochastical uncertainty is characterised by a known effect (for example, the strength of a structural component varies from part to part within a production series, so the weakest ones might fail during usage) and an acceptably quantified probability [Engelhardt et al. 2012]. The acceptably quantified probability might be given by a probability distribution with characterising parameters. The second type examined is estimated uncertainty, where the effect is also known, but the probability of its occurrence is only partially quantified [Engelhardt et al. 2012]. An example of estimated uncertainty is when the lowest and highest strengths of the structural component mentioned above are known for a production batch but not how the strength is distributed statistically. Often the designer faces estimated uncertainty in product development when only general information about materials or production technologies is available; loads and therefore stresses and strains are uncertain in many cases as well. The designer always has to consider estimated uncertainty in key variables such as those mentioned.

2.3 Example product

As an example product two beams should be used, both of them with the same geometrical and material properties, with one used as a buckling beam, according to Euler (Section 4.), and one as a simple tension bar. Both have a movable bearing and a fixed bearing. The material is the same over the whole range. Since only the critical load and the strain are of interest, it is sufficient to specify its Young's modulus; strength is not taken into account, but being size-related, it might be subject of further research. The properties for the basic design as well as for the largest (geometrical step size of ten) and smallest step sizes (geometrical step size of 0.1) are shown in Table 1. They follow a Gaussian distribution, except for Young's modulus, which has a rectangular distribution.

Table 1. Product properties

Product property	Smallest size	Basic size	Largest size
Diameter d	0.4 mm ± 0.046 mm	4 mm ± 0.1 mm	40 mm ± 0.22 mm
Length l	10 mm ± 0.46 mm	100 mm ± 1 mm	1,000 mm ± 2.15 mm
Young's modulus E	70 kN/mm² ± 2 kN/mm²	70 kN/mm² ± 2 kN/mm²	70 kN/mm² ± 2 kN/mm²
Tensional force F (only tension bar)	10 N ± 0.5 N	100 N ± 5 N	1,000 N ± 50 N
Geometrical step factor	0.1	1	10

3. Goal

Laws of growth are commonly used in type series development. The product properties are calculated only for nominal values. This approach is insufficient, given that using the same production technology usually results in larger relative deviations in downscaling processes or smaller ones in upscaling processes [Wittel et al. 2013]. This is given by the following empirical equation, which describes the tolerance factor i, a proportional factor for tolerances [Wittel et al. 2013]:

$$i = 0.45 \cdot \sqrt[3]{d} + 0.001 \cdot d \qquad (5)$$

These deviations from nominal values (also possible for mechanical and thermal loads) can be addressed as uncertainty. The result of not considering this uncertainty properly is a product that may not be safe enough (mostly when downscaling) or that is more expensive due to unnecessary height tolerances (mostly when upscaling). Working with relatively small diameters, the second addend is small compared to the first. If written using step sizes as the basic value, the deviations of product properties can be described as

$$\phi_i = {\phi_d}^{1/3} = {\phi_l}^{1/3} \qquad (6)$$

This law of growth leads to nonlinear growth of uncertainty due to technology initiated deviations. In this case, uncertainty is understood as the relative variance $Var(X_i)/X_i$ of the target parameter. For the example product mentioned above, X represents the critical load or the mechanical strain. This uncertainty can be classified as stochastic uncertainty if probability distributions for the product properties are given [Hanselka et al. 2010]. If only intervals are available, estimated uncertainty exists [Hanselka et al. 2010].

With this information, an approach for analysing size-dependant uncertainty is given (Figure 1). The approach differentiates between two scenarios, one in which estimated uncertainty, like product properties that are realized within an interval, has to be faced but no further information except an estimated maximum and minimum value is available. In this case, scenario-based laws of growth can be used to describe the uncertainty of the targeted product properties. If the designer knows probability distributions of input parameters of a problem, it is possible to do a Monte Carlo simulation in order to predict the probability distribution of the targeted product properties.

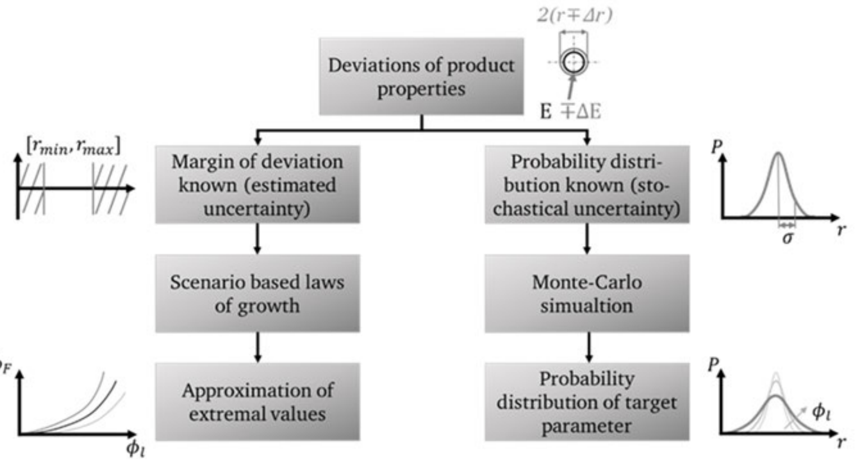

Figure 1. Approach for handling different kinds of size-dependent uncertainty

4. Laws of growth for best case and worst case scenarios

Depending on information available to the designer, a common approach is to develop worst-case and, if it is helpful, best-case scenarios. As mentioned above, a worst case and a best case can be defined for the target parameter. For these two cases, a calculation of the estimated product properties is performed and compared to the planned (nominal) value. Basic inputs for the mentioned example products are length, diameter and Young's modulus. Their deviations are calculated according to the proposition in Section 3. With the minimum critical load given by

$$F_{crit,min} = \frac{E_{min} \cdot \pi^3 r_{min}^4}{4 l_{max}^2} \tag{7}$$

the law of growth for the minimum critical load is

$$\phi_{F\,crit,min} = \frac{\left(l_{nom,0} + \Delta l_0\right)^2 \left(r_{nom,0} \cdot \phi_l - \Delta r_0 \cdot \phi_l^{1/3}\right)^4}{\left(r_{nom,0} - \Delta r_0\right)^4 \left(l_{nom,1} \cdot \phi_l + \Delta l_0 \cdot \phi_l^{1/3}\right)^2} \tag{8}$$

which cannot be written using only step factors. The laws of growth for minimum and maximum critical load (as well as the nominal load) are plotted in Figure 2, as well as the relative deviation from the nominal critical load.

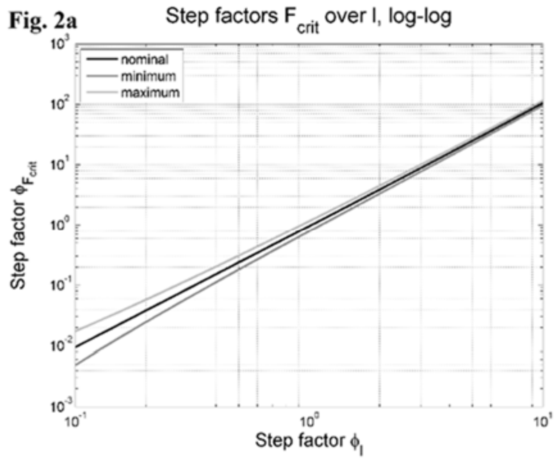

Figure 2a. Law of growth for the buckling beam

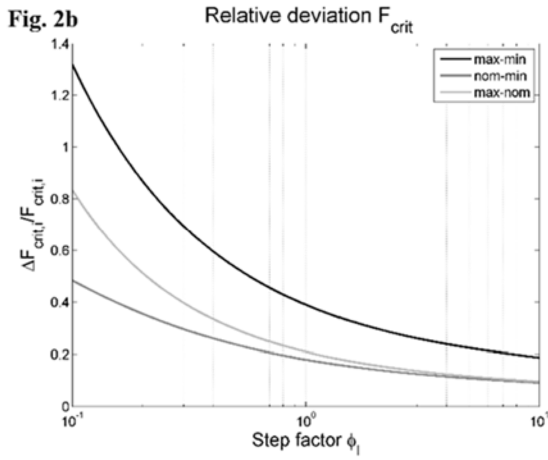

Figure 2b. Margin of deviation of critical force relative to the nominal scaled critical force

This basically describes estimated uncertainty – the boundaries of certain product properties are more or less known but it is not known how they are distributed within those boundaries [Hanselka et al. 2010]. Looking at the critical load's relative deviation (right plot in Figure 2) the very high deviations of the target parameter from the nominal value are a result of the nonlinear growth of production technology-based deviations of the input parameters. In addition to this, the minimum critical load has a significantly higher relative deviation from the system's nominal target parameter value than the maximum critical load. This is a result of the physical correlation of input and target parameters that overemphasise the effect of large relative negative deviation. The diameter has relatively large deviations ($\Delta d/d$) and has influence on the system's behaviour by being used in the calculation by the fourth power. The second product that was examined confirms this: the tension bar with the same product properties. If mechanical strain is the target parameter, it is calculated with the second power of the diameter, while Young's modulus has the same linear influence as for the critical load of the buckling beam, and length is not taken into account at all. Another effect of the lower sensitivity of the tension bar's law of growth is that the relative deviation from the planned product properties is smaller than the buckling beam when scaled down. From one specific point onwards the deviation is larger than in the system with higher sensitivity to deviations of the parameters. This is also a result of nonlinear growth of deviations in product properties caused by production and can be seen in Figure 3.

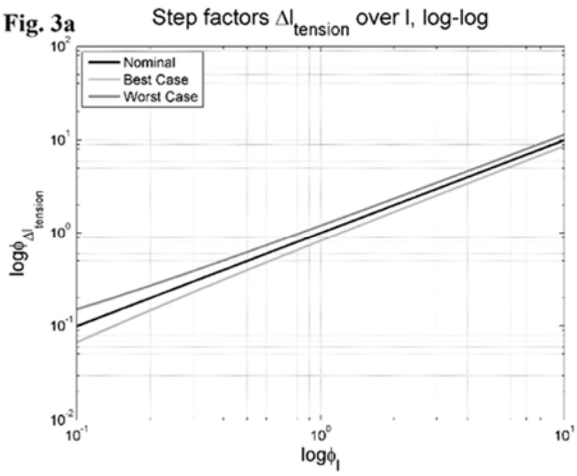

Figure 3a. Law of growth for tension

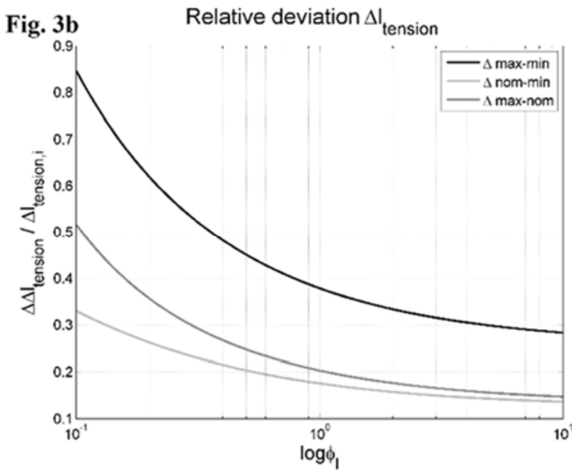

Figure 3b. Margin deviation of its elongation relative to the nominal scaled elongation

Creating best and worst-case scenarios helps to understand how the deviations of the target parameter will develop when scaling the product up or down, in nonlinear and specific laws of growth for product properties.

5. Monte Carlo simulation for scaled products

As the development of a product moves on, more and more information becomes available to the designer. Even the probability distributions related to production processes might become known. Having access to this information means that the uncertainty can be classified as stochastical uncertainty. The following section focuses on the statistical behaviour that the target parameter shows.

5.1 Assumptions

To get started, assumptions about the stochastical properties of the technology-based deviations of product properties have to be set. First, the law of growth for the deviations shall be the same as for the best-case/worst-case scenarios, namely Equation 6. Besides this, the standard deviation σ of the probability deviation shall grow according to this law of growth:

$$\phi_\sigma = {\phi_l}^{1/3} \quad (9)$$

The standard deviation shall be 1/6 of the uncertainty (deviation), as shown in Table 1, which is a common goal in quality management [Toutenburg and Knöfel 2009]. The scaled product property shall have the same kind of probability distribution (similarity between the probability distributions of two sizes). Figure 3 shows these assumptions. For illustration purposes, the deviations and the uncertainty have been increased compared to the uncertainty of the product property 'length' used before (only for Figure 4; all other figures and outcomes are based on Table 1).
The following probability distributions were chosen for the input parameters:

- Diameter *d*: Gaussian probability distribution, σ_d is 1/6 of the planned maximum deviation
- Length *l*: Gaussian probability distribution, σ_l is 1/6 of the planned maximum deviation
- Young's Modulus *E*: Rectangular probability distribution, symmetrical to nominal (mean and median) value
- Force *F* (tension, tension bar only): Gaussian probability distribution, σ_F is 1/6 of the planned maximum deviation

These assumptions are taken into account in all calculations in the following Monte Carlo simulation.

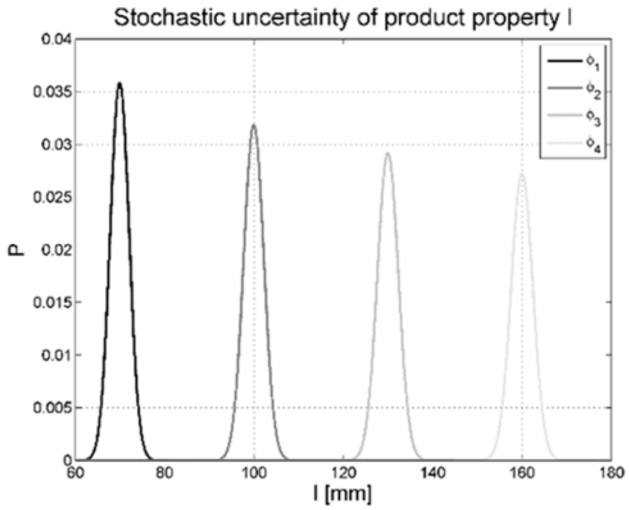

Figure 4. Nonlinear growth of uncertainty in one product property

5.2 Monte Carlo simulation

It is useful to simulate a large variety of parameter combinations with a Monte Carlo simulation to find out what the real input parameter-determined distribution of the target parameter is and how its probability distribution develops while scaling up or down. A convolution of the probability distributions of the input parameters is insufficient due to the different kinds of probability distribution that can possibly be taken into account. Since the following approach should also be suitable for use in problems of a different kind, probability distributions that cannot be convoluted as easily as Gaussian probability distributions can be covered by the Monte Carlo simulation approach. One example is a Weibull probability distribution for scaling problems that involve material or endurance strength as well as system reliability [Haibach 2006], [Pham 2006].
In order to get a result for this problem a large number of calculations has to be performed for the example products. For every calculation a random combination of values generated according to their individual stochastical frequency of occurrence is used as input parameters. The range of sizes between step factor 0.1 and 10, as specified in Table 1, is split up into 1,000 equally distanced steps of scaling. For every one at least 100,000 samples of input parameters are generated (up to 500,000 for validation purposes), according to the assumptions specified in Section 5.1. A calculation of the underlying physical law then has to be performed to obtain the output parameter (critical load for buckling of a beam or tensional strain of the bar). The results (one set of at least 100,000 calculated possible values of the output parameter) are analysed for every scaling factor, and the standard deviation and mean value are calculated and a histogram is generated.

5.3 Analysing the results of the Monte Carlo simulation

A Monte Carlo simulation generates information that has to be analysed in order to give recommendations on how to deal with uncertainty when scaling a product. This is done by having a closer look at the parameters that describe the resulting probability distribution.

Mean Value

The mean value is basically the same as the nominal value for both of the example products and over the whole size range. The gap between nominal value and mean value is under 1% for both products, having its maximum at the smallest step factor and decreasing monotonically. This validates the use of laws of growth for scaling up or down the nominal value as a representative of the mean value.

Standard deviations

The standard deviation grows more slowly than geometric size (due to Equation 6, because the standard deviation of the target parameter depends on standard deviations of the input parameters). This means the relative standard deviation σ_i/σ_0, which refers to the standard deviation of the target parameter of the basic sized product, decreases. Uncertainty, if understood as a margin of deviation, decreases with scaling up and increases with scaling down. A law of growth for standard deviation can be approximated using a polynomial or a representative exponential.

Skewness and kurtosis

The next step is to interpret the histograms. For both example products, they are asymmetric, which means they have skewness. For small step factors (downscaling) skewness increases. This is a result of the nonlinearity of the underlying physical law: if the deviations of the production process get relatively large compared to the parameter they are based on (e.g. Δd to d), the nonlinearity of the physical law generates a very large deviation of the target parameter. Due to the nonlinearity of the physical law which links input parameters to the target parameter, the probability distribution of the target parameter has a noticeable skewness (0.18) when large relative deviations in production processes are present (Figure 5a). This corresponds with small step factors. The skewness is also linked to the underlying physical law: the higher the sensitivity to parameters with large relative deviation the higher the skewness.
The excess kurtosis of this depends strongly on the size of the product. Scaling results in a platykurtic (negative excess kurtosis) behaviour for small step factors (Figure 5a) while having leptokurtic behaviour (positive excess kurtosis) at large step factors (Figure 5c). The distribution of the unscaled product is shown in Figure 5b.

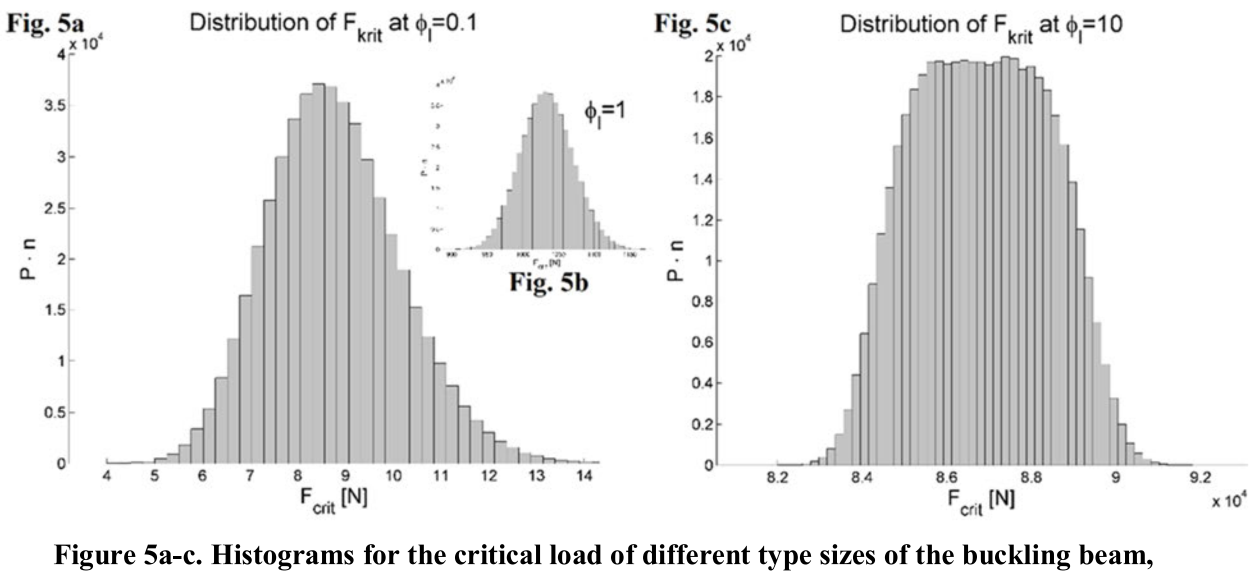

Figure 5a-c. Histograms for the critical load of different type sizes of the buckling beam, showing change in skewness and kurtosis

This means that the target parameter is distributed in a more even way over its margin of deviation when scaled up, whereas it concentrates around the mean value when scaled down (while having a larger relative margin of deviation). The explanation for this system behaviour lies in the rectangular distribution of Young's modulus. Having small diameters results in large relative geometric tolerances because of their nonlinear growth. This means that the relative deviation of the target parameter is also huge and mostly depends on the geometric tolerances. In the opposite direction of scaling, large beams have small relative geometric tolerances and therefore this does not affect the target parameter much. In this case, most of the target parameter's deviation is caused by the deviation in material properties (in this case, stiffness).

Influence of the growth exponent of production initiated deviations of product properties

The growth exponent of the deviations caused by production processes also has an important impact on how uncertainty of dependent product properties develops while scaling a product. As mentioned previously, a widely used law of growth uses the exponent 1/3 (Equation 6) but production technology-based deviations in product properties grow according to another law. It is possible that they do not increase at all over the whole size range needed, resulting in an exponent of zero. Perhaps they grow in proportion with geometric size; maybe they grow with an exponent larger than one due to exceeding of the production process capabilities. It is useful to know how uncertainty will react to changing the law of growth for those deviations. All assumptions made in Section 5.1 are still valid in the following analysis.

Having a growth exponent of zero results in the same behaviour described in previous paragraphs. Just the effect of having very large relative tolerances when scaling down to small step factors or having very small relative tolerances when scaling up is stronger. Excess kurtosis and skewness reach larger values at extreme step factors; the relative standard deviation is larger at small step factors and smaller at large step factors.

A growth exponent of one results in constant excess kurtosis and skewness, with small values for the example products, as specified in Table 1. The relative standard deviation is constant for the whole type range. All calculation can be performed with nominal laws of growth for the target parameter and multiplication with a constant factor for maximum and minimum value.

Using a very bad production process with deviations growing with an exponent of growth larger than one results in a reversed effect: the kurtosis gets platykurtic when scaled up whereas the skewness increases, and the relative standard deviation increases while scaling up and decreases while scaling down.

6. Conclusions, and comparison between scenario laws of growth and Monte Carlo simulation

This paper introduced two ways to handle uncertainty when designing a type range. Nonlinear growth effects due to deviations in product properties are examined in particular. Technological effects cause nonlinearity during production. Both ways use laws of growth as a common method to develop a type range from one basic-sized product.

For relatively early design stages, a scenario-based approach is found using best-case and worst-case scenarios for combining input parameters. Extreme product properties (output parameters) for each nominal step size are calculated this way. These two scenarios are translated into laws of growth. Every scenario leads to one law of growth for one extreme value.

In later design stages, more information about the production process might be available to the designer. If information about the probability distributions of input parameters of the basic product is available and production technology is the same across the whole type range, using Monte Carlo simulation gives a tool for predicting the output product properties. Based on this information, a law of growth for the resulting standard deviation of the output parameter (targeted product property) can be approximated. The type of probability distribution varies over the type range. Scaling a product makes it necessary to examine the skewness, kurtosis and deviation of the probability distributions realised in the target product property.

Comparing the numerical values of these two methods, there is another important fact to be taken into account during the product development process. There is one point (a sole step factor) in product scaling where best-case/worst-case scenarios and Monte Carlo simulation results match. Using other step factors results in underestimating or overestimating uncertainty using scenario laws of growth if the margin of deviation for estimated uncertainty is the same as for the targeted standard deviation for stochastical uncertainty that is acceptable in this product property, e.g. 6σ (Figure 6).
The step factor with which the best-case/worst-case scenario changes place with 6σ plots varies depending on the relative deviation of the input factors. Smaller relative deviations of the input parameters move the crossing points to larger step factors. Depending on whether the output parameter should be higher or lower for the best-case scenario, the designer might use insufficient safety margins using best or worst-case scenarios, or might adopt unnecessarily high safety margins. As Figure 6 shows, those deviations can easily reach about 10% of the scenario-based values.

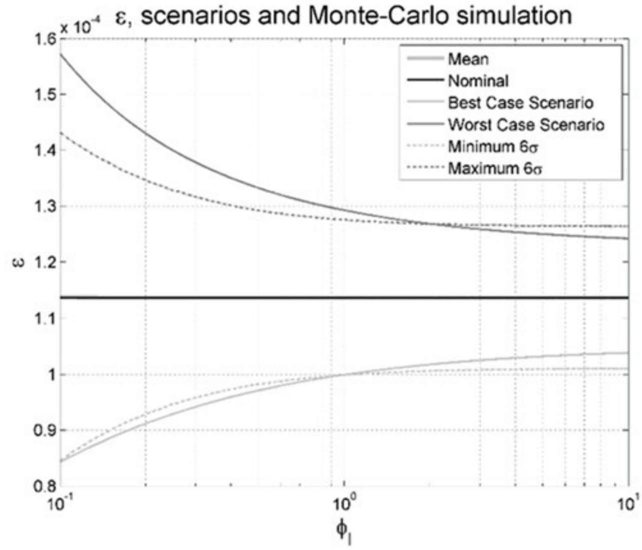

Figure 6. Comparison of scenario-based laws of growth and Monte Carlo simulation

7. Outlook

There are several options for avenues of future research. An examination of other types of laws of growth, like polynomial laws of growth instead of monomial ones, is important as cost growth usually is split into elements of first, second or third degree (relative to the step factor of geometrical scaling) and constant elements [Most 1989]. Besides polynomial laws of growth, the behaviour of uncertainty in scaled complex systems is of huge interest. The link of scaled parts in such systems has to be examined too. An example product would be a truss made out of the beams, as mentioned above.
The use of scenario-based laws of growth contains the risk of insufficient estimation of product properties, as shown in Section 3. These risks might be reduced if fuzzy numbers are used. In early design stages, the designer might have enough information to use fuzzy intervals or probability distributions that can be approximated with fuzzy sets but not enough to use precise probability distributions. In both cases, using fuzzy sets might help the designer to approximate scaled uncertainty in a more precise way than using laws of growth with singular values to describe the scenarios used.
Finally, methods for scaling products under the influence of uncertainty as well as measuring the uncertainty caused by scaling methods should be examined.
In the long-term, a methodology for handling uncertainty while developing type ranges will be developed to help designers avoid unnecessarily high safety margins or taking risks that are higher than currently estimated.

Acknowledgement

We would like to thank the Deutsche Forschungsgemeinschaft (DFG) for founding this project within the Collaborative Research Centre (CRC) 805.

References

Chen, C. Q., Shi, Y., Zhang, Y. S., Zhu, J., Yan, Y. J., "Size Dependence of Young's Modulus in ZnO Nanowires", Physical Review Letters 96, The American Physical Society, 2006, pp. 075505-1-075505-4.

Engelhardt, R. A., Koenen, J., Brenneis, M., Kloberdanz, H., Bohn, A., "An Approach to Classify Methods to Control Uncertainty in Load-Carrying Structures", Applied Mechanics and Materials Vol. 104, 2012, pp. 33-44.

Gibbings, J. C., "Dimensional Analysis", Springer-Verlag London, UK, 2011.

Gross, D., Hauger, W., Schröder, J., Wall, W. A., "Technische Mechanik. Band 2: Elastostatik", Springer-Verlag Berlin, 2012.

Haibach, E., "Betriebsfestigkeit. Verfahren und Daten zur Bauteilberechnung", Springer-Verlag Berlin, 2006.

Hanselka, H., Platz, R., "Ansätze und Maßnahmen zur Beherrschung von Unsicherheit in lasttragenden Systemen des Maschinenbaus", Journal Konstruktion, VDI-Verlag Düsseldorf, Nov./Dec. 2010, pp. 55-62.

Most, E., "Mathematische Verfahren und Hilfsmittel bei der Anwendung von Kostenwachstumsgesetzen für ähnliche Konstruktionen", VDI-Verlag Düsseldorf, 1989.

Pahl, G., Beitz, W., Feldhusen J., Grote, K. H., "Konstruktionslehre. Grundlagen erfolgreicher Produktentwicklung. Methoden und Anwendung", Springer-Verlag Berlin, 2007.

Pahl, G., Rieg, F., "Kostenwachstumsgesetze für Baureihen: mit Anwendungsbeispielen und Rechnerprogrammen für die Konstruktionspraxis", VDI-Verlag Düsseldorf, 1984.

Pham, H. (ed.), "Springer Handbook of Engineering Statistics", Springer-Verlag London, UK, 2006.

Röhlig, C.-C., "Evaluierung elastischer Eigenschaften von Mikro- und Nanostrukturen", diss., Technische Fakultät der Albert-Ludwigs-Universität Freiburg im Breisgau, 2011.

Toutenburg, H., Knöfel, P., "Six Sigma. Methoden und Statistik für die Praxis", Springer-Verlag Berlin, 2009.

Wiebel, M., Engelhardt, R. A., Habermehl, K., Birkhofer, H., "Uncertainty in Process Chains and the Calculation of Their Propagation via Monte-Carlo simulation", Proceedings of the 12th International Dependency and Structure Modelling Conference, DSM'10, Cambridge, UK, 2010.

Wittel, H., Muhs, D., Jannasch, D., Voßiek, J., "Roloff/Matek Maschinenelemente. Normung, Berechnung, Gestaltung", Springer Vieweg, Wiesbaden, 2013.

Julian Lotz, M.Sc, Research Associate
Technische Universität Darmstadt, pmd
Magdalenenstr. 4, 64289 Darmstadt, Germany
Telephone: +49 6151 16 4399
Telefax: +49 6151 16 3355
Email: lotz@pmd.tu-darmstadt.de
URL: http://www.pmd.tu-darmstadt.de