9 MATHEMATICAL MODELS FOR DESIGNING VEHICLES FOR RIDE COMFORT

Manoj K. Mahala[a], Prasanna Gadkari and Anindya Deb[b]

CPDM, Indian Institute of Science, Bangalore India, Tel: +91-(0)80-22933136.
E-mail: [a]manoj@cpdm.iisc.ernet.in, [b]adeb@cpdm.iisc.ernet.in

Various mathematical models, also known as lumped parameter models, have been employed for studying vehicle dynamics. In the present study, the responses of different models are studied and compared for different road conditions. The mathematical models considered are: (i) a two degrees-of-freedom system (quarter car model), (ii) a four degrees-of-freedom system (half car model) and (iii) a seven degrees-of-freedom system (full car model). These systems are analyzed using SIMULINK®for pitch and roll modes of inputs. Various response parameters such as body acceleration and body displacement are obtained using representative passive suspension system properties. The current study aims at evaluating the capabilities of different modeling approaches mentioned above and their deficiencies.

Keywords: Lumped Parameter Modeling, Vehicle Dynamics, Mathematical Modeling, Passive Suspension System, Pitch, Roll.

1. INTRODUCTION

Lumped parameter models are commonly used for assessing performance of different dynamic system, especially in automotive research. Researchers have used lumped parameter models in vehicle safety,[1−3] modeling of powertrain to analyse effect of shifting behavior on vehicle handling[4], suspension system performance study,[5−7] development and analysis of active suspension system,[8−16] etc. The current study covers suspension system modeling using lumped parameter models of increasing degrees of complexity such as 2 degrees-of-freedom system (quarter-car model), four degrees-of-freedom system (half-car model) and seven degrees-of-freedom system (full-far model). Deb *et al.* have used lumped parameter modeling in front impact test analysis[1] and headform impact testing.[2,3] Shim and Zhang[4] have studied effect of powertrain transient shifting on vehicle handling using a 14 degrees-of-freedom vehicle model and lumped-mass model of power train. Rehan, *et al.*[5] studied suspension system behavior of a tramcar by varying suspension stiffness using a full-car model. A full-car model has also been used by Kim and Ro[6] for describing model reducing techniques. A half-car model has been developed by Gao, Zhang and Du[7] for analyzing vehicle dynamics with random parameters. Mehra *et al.*[8] have used quarter-car model to develop model preview control for active suspension system. Hong, Jeon and Sohn[9] have focused on rotational movement of unsprung mass and developed an optimal pole-placement control for McPherson Strut Suspension System. Hung-Yi and Huang[10,11] have used a quarter-car for designing model-free adaptive sliding controller. Half-car vehicle models have been used for design of constrained H_∞ active suspension system by Chen, Liu and Liu[12] and in design of tandem active-passive suspension system by Giua, Seatzu and Usai.[13] A full-car model has been employed in development of an active suspension control by combining a filtered feedback control scheme and an “input decoupling transformation” by Ikenaga *et al.*[14] Smith and Wang[15] have applied controller parameterization to vehicle active suspension using 2, 4 and 7 degrees-of-freedom models.

In the current study, the responses of different models are studied. The mathematical models considered are: (i) a two degrees-of-freedom system (quarter-car model), (ii) a four degrees-of-freedom system (half-car model) and (iii) a seven degrees-of-freedom system (full-car model). These models

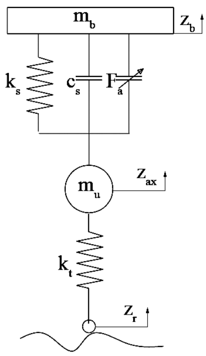

Figure 1. 2-DOF model.

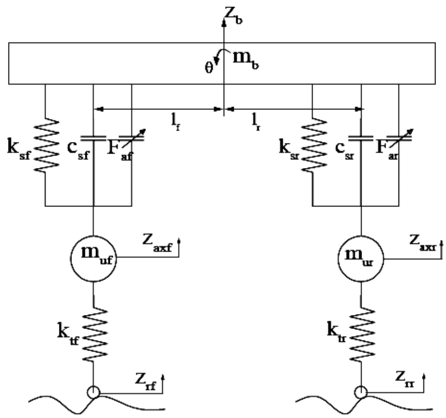

Figure 2. 4-DOF model.

are compared for pitch and roll modes of inputs to tyre. These systems are solved using SIMULINK® and response parameters such as body acceleration, body displacement, etc. are obtained using representative passive suspension system properties. The current study has thrown light into the capabilities of different modeling approaches mentioned above and their deficiencies.

2. QUARTER, HALF AND FULL CAR MODEL:

A two degrees-of-freedom system, quarter car model of suspension system is shown in Figure 1. It represents the suspension system at any of the four wheels of the vehicle and the degrees-of-freedom are displacement of axle and displace of the vehicle body at the particular wheel. The model consists of a spring k_s, a damper c_s, and an active force actuator F_a.The active force F_a is said to be zero in a passive suspension. The sprung mass m_b and the unsprung mass m_u, represent the mass of the vehicle body equivalent to quarter-car and equivalent mass of the axle and tire respectively. The spring k_t represents the vertical stiffness of the tire. The vertical displacements of the sprung mass, unsprung mass and road profile are represented by z_b, z_{ax} and z_r respectively in static equilibrium.

A four degrees-of-freedom system, half car model of suspension system is shown in Figure 2. The model represents the pitch and heaves motions (θ and z) of the vehicle body and the vertical translation of the front and rear axles (z_{axf} and z_{axr}). The unsprung masses at front and rear axles are represented by m_{axf} and m_{axr} respectively. The sprung mass of vehicle body is represented by m_b and its moment of inertia about pitch axis is I_θ. Symbols l_f and l_r represent distance of front and rear axle from centre of gravity of the vehicle body.

The seven degrees-of-freedom in full car model are the heave z, pitch θ and roll ϕ of the vehicle body and the vertical motions (z_{axfR}, z_{axrR}, z_{axfL} and z_{axrL}) of each of the four unsprung masses (m_{axfR}, m_{axrR}, m_{axfL} and m_{axrL}). Other parameters such as vehicle body mass, stiffness of suspension are represented using m_b, k_{sfR}, k_{srR}, k_{sfL} and k_{srL}, whereas stiffness of tires are represented by k_{tfR}, k_{trR}, k_{tfL} and k_{trL}. The model is shown in Figure 3 where only fR suffix variables are labeled.

2.1. Mathematical Models:

The equations of motion for each model can be written by free body diagram concept. The equations for the 2 degrees-of-freedom system, Figure 1, (quarter-car model) are as following:

$$m_b\ddot{z}_b = -k_s(z_b - z_{ax}) - c_s(\dot{z}_b - \dot{z}_{ax}) + F_a \tag{1}$$

$$m_u\ddot{z}_{ax} = k_s(z_b - z_{ax}) + c_s(\dot{z}_b - \dot{z}_{ax}) - F_a - k_t(z_{ax} - z_r) \tag{2}$$

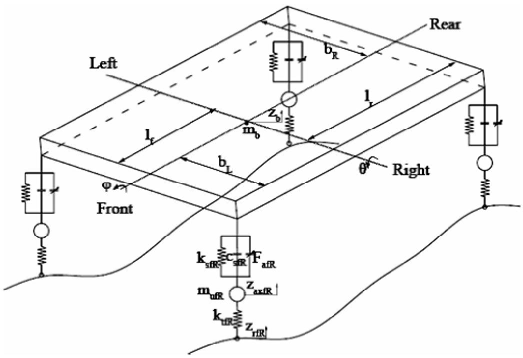

Figure 3. 7-DOF Model (3^{rd} suffices R and L for right and left and 2^{nd} suffices f and r for front and rear).

In standard second-order matrix form, the system can be represented as follows:

$$M\ddot{z} + C\dot{z} + Kz = Az_r + BF_a \tag{3}$$

or

$$\begin{bmatrix} m_b & 0 \\ 0 & m_u \end{bmatrix}\begin{Bmatrix} \ddot{z}_b \\ \ddot{z}_{ax} \end{Bmatrix} + \begin{bmatrix} c_s & -c_s \\ -c_s & c_s \end{bmatrix}\begin{Bmatrix} \dot{z}_b \\ \dot{z}_{ax} \end{Bmatrix} + \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix}\begin{Bmatrix} z_b \\ z_{ax} \end{Bmatrix} = \begin{Bmatrix} 0 \\ k_t \end{Bmatrix} z_r + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} F_a \tag{4}$$

Where

$$M = \begin{bmatrix} m_b & 0 \\ 0 & m_u \end{bmatrix} K = \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix}$$

and the other matrices are defined in equation.

Similarly the equations of motion for 4 degrees-of-freedom system (half-car model) shown in Figure 2 are:

$$m_b\ddot{z}_b = -k_{sf}(z_{bf} - z_{axf}) - c_{sf}(\dot{z}_{bf} - \dot{z}_{axf}) + F_{af} - k_{sr}(z_{sr} - z_{axr}) - C_{br}(\dot{z}_{axr}) + F_{ar} \tag{5}$$

$$m_{uf}\ddot{z}_{axf} = k_{sf}(z_{bf} - z_{axf}) + c_{sf}(\dot{z}_{bf} - \dot{z}_{axf}) - F_{af} - k_{tf}(z_{axf} - z_{rf}) \tag{6}$$

$$m_{ur}\ddot{z}_{axr} = k_{sr}(z_{br} - z_{axr}) + c_{sr}(\dot{z}_{br} - \dot{z}_{axr}) - F_{ar} - k_{tr}(z_{axr} - z_{rr}) \tag{7}$$

$$I_\theta\ddot{\theta} = l_r(-k_{sr}(z_{br} - z_{axr}) - C_{sr}(\dot{z}_{br} - \dot{z}_{axr}) + F_{ar}) - l_f(-k_{sf}(z_{bf} - zaxf) - C_{sf}(\dot{z}_{bf} - \dot{z}_{axf}) + F_{af}) \tag{8}$$

$$\theta = \frac{(z_{bf} - z_{br})}{l_f + l_r} \tag{9}$$

Where

And, similarly 7 degrees-of-freedom system (full-car model) and 4 degrees-of-freedom system for roll simulation are modeled. Schematic diagram of full-car model is shown in Figure 3.

2.2. Vehicle Parameters and Input Road Conditions:

In the current study, vehicle parameters are taken from Rehan *et al.*[5] and are mentioned in Table 1. And the equivalent parameters for half-car model and quarter-car model are given in Table 2 and Table 3 respectively.

Road condition input is shown in Figure 4 below. Solid line in the plot is pulse of 0.07 m height for duration of 0.08 seconds which is used for pitch mode simulation where as dashed line shows pulse used for simulating roll mode of the vehicle which has same amplitude as pitch mode pulse and its duration is 0.38 seconds.

Table 1. Vehicle specifications for full-car model.

Parameters	Vehicle
Sprung Mass (kg)	1354
Unsprung Masses (kg)	59
Tyre Stiffness (N/m)	190000
Suspension Stiffness (N/m)	15000
Suspension Damping (N/(m/sec))	450* & 1070**
Roll Axis Moment of Inertia (kg-m^2)	729.67
Pitch Axis Moment of Inertia (kg-m^2)	1406.895
Front Tyre – C.G. Distance (m)	1.784
Rear Tyre – C.G. Distance (m)	1.633
Right Tyre – C.G. Distance (m)	0.805
Left Tyre – C.G. Distance (m)	0.795

*, **: Suspension damping is taken different for negative and positive velocity.
Source Rehan *et al.*[5]

Table 2. Vehicle specifications for half-car model.

Parameters	Vehicle
Sprung Mass (kg)	677
Unsprung Masses (kg)	59
Tyre Stiffness (N/m)	190000
Suspension Stiffness (N/m)	15000
Suspension Damping (N/(m/sec))	450* & 1070**
Pitch Axis Moment of Inertia (kg-m^2)	1406.895
Front Tyre – C.G. Distance (m)	1.784
Rear Tyre – C.G. Distance (m)	1.633

*, **: Suspension damping is taken different for negative and positive velocity.

Table 3. Vehicle specifications for quarter-car model.

Parameters	Vehicle
Sprung Mass (kg)	338.5
Unsprung Mass (kg)	59
Tyre Stiffness (N/m)	190000
Suspension Stiffness (N/m)	15000
Suspension Damping (N/(m/sec))	450* 1070**

*, **: Suspension damping is taken different for negative and positive velocity.

3. RESULTS

Displacement and acceleration of sprung mass are compared with the results of Rehan *et al.* in pitch mode and roll mode. These comparisons are shown in Figure 5, Figure 6, Figure 7 and Figure 8. These comparisons validate the model.

Figure 9, Figure 10, Figure 11 and Figure 12 show the comparison of 2-DOF and 4-DOF models with 7-DOF model in pitch and roll modes. Figure 9 and Figure 10 show that in pitch mode 4-DOF model matches well with 7-DOF model but 2-DOF model does not. And Figure 11 and Figure 12 show that in roll mode neither 4-DOF model nor 2-DOF model give good results. Since 2-DOF model and 4-DOF model includes road conditions only for 1 and 2 tyres respectively, so these can not reflect the effect of road conditions at remaining tyres. For example in pitch mode, when front tyres go over bump, rear tyres stay on ground which causes reduction in the amplitudes of displacement and acceleration of the body. Then as rear tyres reach to bump, front tyres are on ground which also causes reduction in amplitudes. But, in 2-DOF model, input road conditions at other tyres can not be incorporated. Similarly in roll

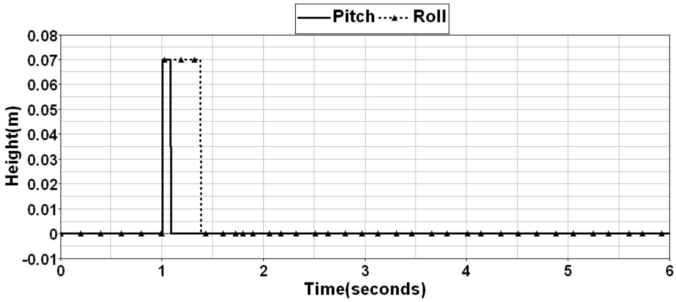

Figure 4. Pitch mode pulse (solid line) and roll mode pulse (dashed line).

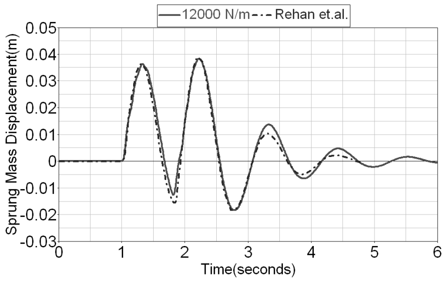

Figure 5. Sprung mass displacement comparison of 7-DOF model Rehan *et al.*[5] results in pitch mode.

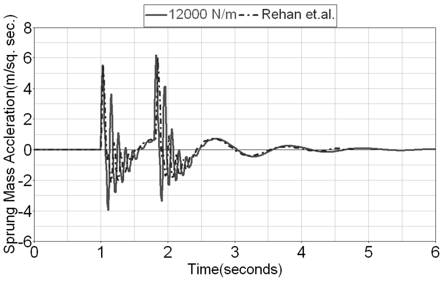

Figure 6. Sprung mass acceleration comparison of 7-DOF model and Rehan *et al.*[5] results in pitch mode.

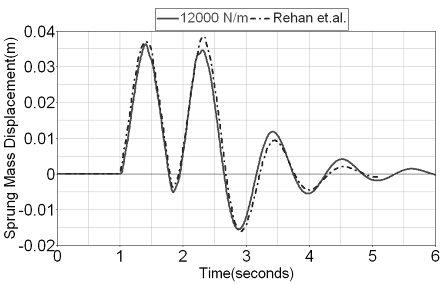

Figure 7. Sprung mass displacement comparison of 7-DOF model rehan *et al.*[5] results in roll mode.

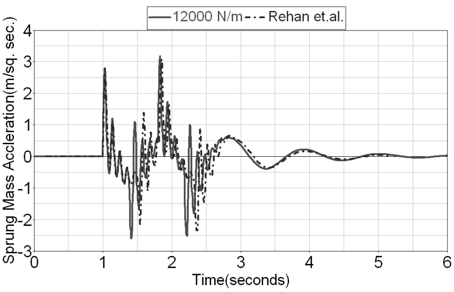

Figure 8. Sprung mass acceleration comparison of 7-DOF model rehan *et al.*[5] results in roll mode.

mode simulation with 4-DOF model road conditions at rear tyres can not be incorporated. A model which does not include road conditions at all the tyres, results in similar pattern of displacement and acceleration with different amplitude and does not show the effect of input conditions which are not included in the simulation. For example road input at rear tyres can not be incorporated in roll mode simulation using 4-DOF model.

Pitch mode simulation results with different suspension stiffness values are shown in Figure 13, Figure 14, Figure 15 and Figure 16 using 7-DOF model and 4-DOF model respectively. These results show that the maximum amplitudes of displacement and acceleration of sprung mass decrease with the suspension stiffness but time taken in damping out of the vibration increases.

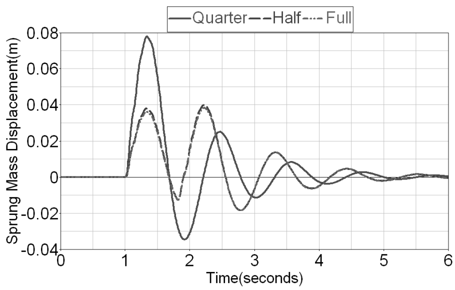

Figure 9. Sprung mass displacement comparison of 2, 4, 7-DOF models in pitch mode simulation.

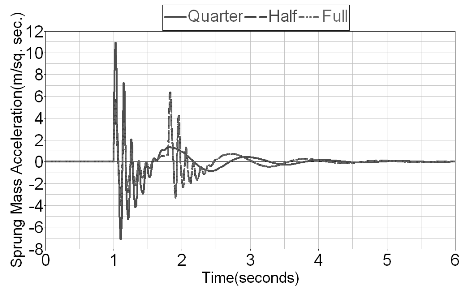

Figure 10. Sprung mass acceleration comparison of 2, 4, 7-DOF models in pitch mode simulation.

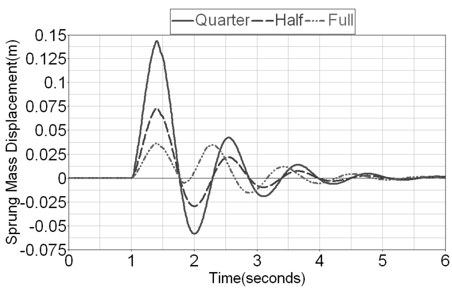

Figure 11. Sprung mass displacement comparison of 2, 4, 7-DOF models in roll mode simulation.

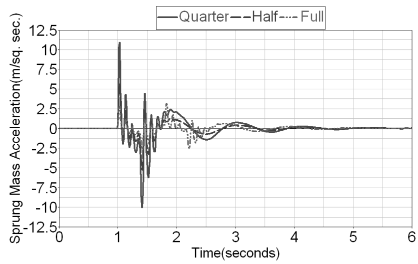

Figure 12. Sprung mass acceleration comparison of 2, 4, 7-DOF models in roll mode simulation.

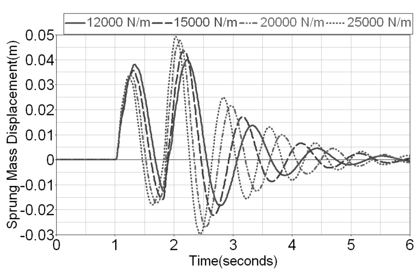

Figure 13. Displacement of sprung mass using 4-DOF model (pitch mode).

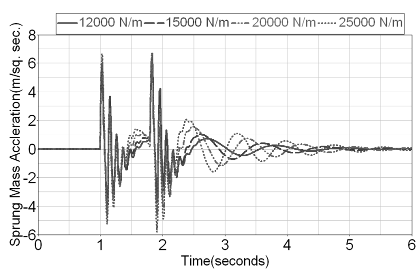

Figure 14. Acceleration of sprung mass using 4-DOF model (pitch mode).

7-DOF model results for roll simulation are shown in Figure 17 and Figure 18 which are also indicating that softer is the suspension, it is better for ride comfort as it is shown in pitch mode simulations earlier.

4. CONCLUSION:

Current study concludes that 7-DOF model can simulate pitch and roll accurately and coupled situation as well. At the same time 4-DOF model is also as accurate as 7-DOF in simulating pitch. Since 4-DOF and 2-DOF models are approximations of suspension system and vehicle body these models can not capture the behavior of vehicle body in all situations. As the current study shows that pattern followed

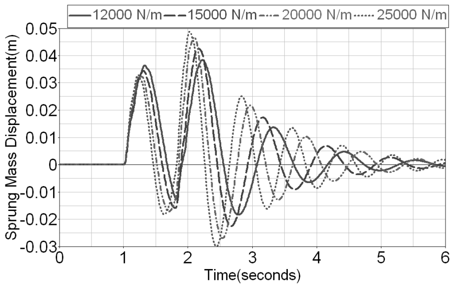

Figure 15. Displacement of Sprung Mass using 7-DOF Model (Pitch Mode).

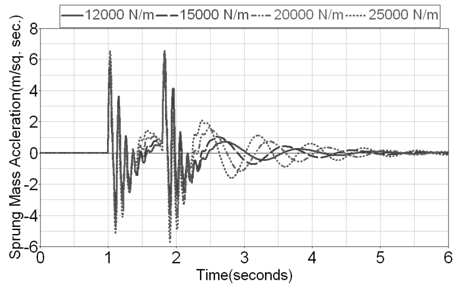

Figure 16. Acceleration of Sprung Mass using 7-DOF Model (Pitch Mode).

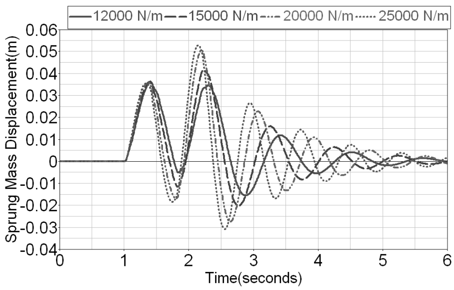

Figure 17. Displacement of sprung mass using 7-DOF model (roll mode).

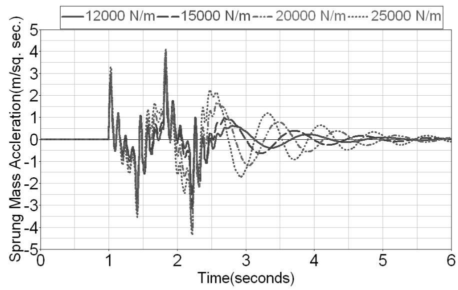

Figure 18. Acceleration of sprung mass using 7-DOF model (roll mode).

by the displacement and acceleration of sprung mass is same for 2, 4 and 7-DOF model so these models can be used for analyzing control algorithms in a crude form. But 2 and 4-DOF models can not capture the behavior of suspension system because road input conditions are not considered at remaining 3 and 2 tyres respectively. Lumped parameter modeling is good for initial studies of different systems but one has to choose suitable modeling procedure and model as well. The study shows that models with higher number of parameters/degrees-of-freedom are better than models with lesser parameters. But higher degrees-of-freedom model requires more data and time to solve it. At the same time in some conditions one can use models with lesser degrees-of-freedom for example pitch mode simulation in the current study.

5. SYMBOLS:

Quarter Car Model:

m_b = Sprung Mass (Equivalent mass of body).
m_u = Unsprung Mass.
z_b = Displacement of m_b.
z_{ax} = Displacement of m_u.
z_r = Road profile.
k_s = Suspension stiffness.
c_s = Suspension damping coefficient.
k_t = Tyre stiffness.

Half Car Model:

m_b = Sprung Mass (Equivalent mass of body).
m_{uf} = Front Unsprung Mass.

m_{ur} = Rear Unsprung Mass.
z_b = Displacement of m_b.
z_{axf} = Displacement of m_{uf}.
z_{axr} = Displacement of m_{ur}.
z_{rf} = Road profile under front tyre.
z_{rr} = Road profile under rear tyre.
k_{sf} = Front suspension stiffness.
k_{sr} = Rear suspension stiffness.
c_{sf} = Front suspension damping coefficient.
c_{sr} = Rear suspension damping coefficient.
k_{tf} = Front tyre stiffness.
k_{tr} = Rear tyre stiffness.
l_f = Distance of C.G. from front tyre.
l_r = Distance of C.G. from rear tyre.
I_θ = Moment of Inertia about pitch axis.
θ = Pitch angle.

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