



NUMERICAL COMPUTATION OF STRESSES, STRAINS AND DISPLACEMENTS IN THE THIN PLATES IN PLASTIC RANGE

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1. Introduction

An incremental formulation of the basic equations of the finite element method is described in this paper. These basic equations are used in stress and strain analysis in structures in which plastic strains have occurred. By numerical analysis of the structure in a non-linear range the external loads are applied in increments. It gives a possibility of keeping track of the plastic zones spreading around the geometrical discontinuity as external loads are increased. The elasto-plastic matrix \mathbf{D}_{ep} was formed for the case of plane stress. The dependence between incremental stress vector and incremental elastic-plastic strain vector is established through matrix \mathbf{D}_{ep} . The elements of the matrix \mathbf{D}_{ep} will not be constant in the case of elastic-plastic state in a structure. They will depend on instantaneous state of stress and strain in a structure.

The paper deals with an analysis of stresses, strains, displacements and spreading of plastic zones in the thin plates of finite dimensions with the geometrical discontinuities in the form of a circular hole, an elliptical hole and a crack. The plates are uniaxially loaded, on stretching in-plane in the direction perpendicular to the big half-axis of the ellipse, or to the crack plane. The holes boundaries are free from loading. The material of the plates is presented schematically as the elastic-perfectly plastic one. The loading is increased uniformly and monotonously, so the local plastic strains occur on the boundaries of the geometrical discontinuities. Consequently, a need for a non-linear analysis arises. Namely, appearance of plastic strains will not immediately cause a plastic collapse of the whole plate. Such an elastic-plastic state of the plate material can be allowed. The function of a plate as a structural element will not be disturbed in that way. Finally, it will have for a consequence a lighter structural element or what is same, permissible loading of a structure will be greater. A possibility of appearance of plastic strains in a structure requires reliable determination of stresses, strains and displacements.

2. Basic equations of the finite element method for structure analysis in an elasto-plastic state

Using an interpolation function, an increase of displacements at any point within the finite element $d\mathbf{f}$ in dependence on an increase of nodal displacements $d\mathbf{U}$ for a particular elements can be written as:

$$d\mathbf{f} = \mathbf{N} d\mathbf{U} \quad (1)$$

where \mathbf{N} is the matrix of shape functions. The strain increment vector $d\mathbf{e}$ within an element is shown through the increase of displacements within the finite element $d\mathbf{f}$, i.e. through the increase of nodal displacements $d\mathbf{U}$ in the following form:

$$d\mathbf{e} = \mathbf{d} d\mathbf{f} = \mathbf{B} d\mathbf{U} \quad (2)$$

Inserting the expression (1) into (2), it is clear that the matrix \mathbf{B} equals \mathbf{dN} , while the matrix \mathbf{d} represents a matrix of the differential operators. The stress increment vector $d\mathbf{s}$ is related to the strain increment vector $d\mathbf{e}$ by the matrix equation:

$$d\mathbf{s} = \mathbf{D} d\mathbf{e} \quad (3)$$

or, by taking into consideration the expression (2), we can write:

$$d\mathbf{s} = \mathbf{DB} d\mathbf{U} = \mathbf{S} d\mathbf{U} \quad (4)$$

where $\mathbf{S} = \mathbf{DB}$ is the so called element stress matrix. The matrix \mathbf{D} in the expression (3) contains elasticity constants of the finite element material. In the case of isotropic, linear elastic material it assumes the form $\mathbf{D} = \mathbf{D}_e$, i. e. \mathbf{D} is equal to the elasticity matrix [Owen 1980]. If behaviour of a material is non-linear, i.e. plastic, then this matrix assumes the form $\mathbf{D} = \mathbf{D}_{ep}$, i.e. it becomes elasto-plastic matrix. Non-linearity of an elasto-plastic problem is noted just in the matrix \mathbf{D} of the rheological constants of a material. A finite element equilibrium equation in arranged form, equals [Owen 1980]:

$$d\mathbf{F} + d\mathbf{F}_{equ} = \left(\int_V \tilde{\mathbf{B}} \mathbf{D} \mathbf{B} dV \right) d\mathbf{U} \quad (5)$$

This equation can be written in a concise form as:

$$d\mathbf{F} + d\mathbf{F}_{equ} = \mathbf{K}_e d\mathbf{U} \quad (6)$$

where \mathbf{K}_e is the element stiffness matrix which is equal to:

$$\mathbf{K}_e = \int_V \tilde{\mathbf{B}} \mathbf{D} \mathbf{B} dV \quad (7)$$

Vector of the equivalent forces in the finite element is obtained by means of a principle of virtual displacements [Pustaic 1990], [Štok 1976] and it amounts to:

$$d\mathbf{F}_{equ} = \int_A \tilde{\mathbf{N}} d\mathbf{q}_e dA + \int_V \tilde{\mathbf{N}} d\mathbf{v}_e dV \quad (8)$$

Similarly, the equilibrium equation for a whole structure can be written in the following form:

$$d\mathbf{R} + d\mathbf{R}_{equ} = \mathbf{K} d\mathbf{U} \quad (9)$$

3. Elasto-plastic matrix \mathbf{D}_{ep} in the case of plane stress

In the case of an elasto-plastic state in a structure the stress increment is connected with the strain increment by a matrix equation [Pustaic 1991], [Pustaic 1990] and [Štok 1976]:

$$d\mathbf{s} = \mathbf{D}_{ep} d\mathbf{e} = \mathbf{D}_e (\mathbf{I} - B^{-1} \mathbf{a} \tilde{\mathbf{a}} \mathbf{D}_e) d\mathbf{e} \quad (10)$$

The expression for an elasto-plastic matrix \mathbf{D}_{ep} is obtained from the above equation:

$$\mathbf{D}_{ep} = \mathbf{D}_e (\mathbf{I} - B^{-1} \mathbf{a} \tilde{\mathbf{a}} \mathbf{D}_e) \quad (11)$$

In this expression \mathbf{I} is a unit matrix of the same order as an elasticity matrix \mathbf{D}_e . The elasto-plastic matrix \mathbf{D}_{ep} can be easily computed if we know the elasticity matrix \mathbf{D}_e , the flow vector \mathbf{a} which yields from the yield criterion and parameter A which yields from the work hardening hypothesis [Owen 1980]. The expression (11) can be represented in the following way:

$$\mathbf{D}_{ep} = \mathbf{D}_e - \mathbf{D}_p \quad (12)$$

\mathbf{D}_p marks the so-called plasticity matrix which is defined by the equation:

$$\mathbf{D}_p = B^{-1} \mathbf{D}_e \mathbf{a} \tilde{\mathbf{a}} \mathbf{D}_e \quad (13)$$

The elasto-plastic matrix \mathbf{D}_{ep} is a symmetrical matrix.

In the case of a plane stress, the stress and strain increment vectors, $d\mathbf{s}$ and $d\mathbf{e}$ respectively, are defined as follows:

$$\begin{aligned} d\tilde{\mathbf{s}} &= [d\mathbf{s}_x, d\mathbf{s}_y, d\mathbf{t}_{xy}] \\ d\tilde{\mathbf{e}} &= [d\mathbf{e}_x, d\mathbf{e}_y, d\mathbf{g}_{xy}] \end{aligned} \quad (14)$$

A deviatoric stress vector is:

$$\tilde{\mathbf{s}} = [s_x, s_y, s_z, s_{xy}] \quad (15)$$

while his components amount:

$$s_x = \frac{1}{3}(2\mathbf{s}_x - \mathbf{s}_y), \quad s_y = \frac{1}{3}(2\mathbf{s}_y - \mathbf{s}_x), \quad s_z = -\frac{1}{3}(\mathbf{s}_x + \mathbf{s}_y), \quad s_{xy} = \mathbf{t}_{xy} \quad (16)$$

It is possible to write the Von Mises yield criterion in the following form:

$$\mathbf{s}_e^2 = \frac{3}{2}(s_x^2 + s_y^2 + s_z^2 + 2s_{xy}^2) = \mathbf{s}_x^2 + \mathbf{s}_y^2 - \mathbf{s}_x \mathbf{s}_y + 3\mathbf{t}_{xy}^2 \quad (17)$$

The elasticity matrix \mathbf{D}_e in the case of plane stress amounts [Owen 1980], [Pustaic 1991], [Pustaic 1990], [Štok 1976]:

$$\mathbf{D}_e = \frac{E}{1-\mathbf{n}^2} \begin{bmatrix} 1 & \mathbf{n} & 0 \\ \mathbf{n} & 1 & 0 \\ 0 & 0 & \frac{1-\mathbf{n}}{2} \end{bmatrix} \quad (18)$$

The flow vector $\mathbf{a} = \partial F / \partial \mathbf{s} = 3\mathbf{s}$ in the case of plane stress assumes the form [Pustaic 1990]:

$$\mathbf{a} = \begin{Bmatrix} \frac{\partial F}{\partial \mathbf{s}_x} \\ \frac{\partial F}{\partial \mathbf{s}_y} \\ \frac{\partial F}{\partial \mathbf{t}_{xy}} \end{Bmatrix} = 3 \begin{Bmatrix} s_x \\ s_y \\ 2\mathbf{t}_{xy} \end{Bmatrix} = \begin{Bmatrix} 2\mathbf{s}_x - \mathbf{s}_y \\ 2\mathbf{s}_y - \mathbf{s}_x \\ 6\mathbf{t}_{xy} \end{Bmatrix} \quad (19)$$

For defining the plasticity matrix \mathbf{D}_p , according to the expression (13) it is necessary to determine the scalar quantity B. The parameter A for the case of plane stress or general three dimensional state of stress is defined as [Owen 1980], [Pustaic 1991], [Pustaic 1990], [Štok 1976]:

$$A = 4H' \mathbf{s}_e^2 \quad (20)$$

where H' is a so-called hardening function which is defined as $H'(\mathbf{e}_e)_p = d s_e / (d \mathbf{e}_e)_p$. This quantity can be determined experimentally from a simple uniaxial yield test. Let us now form the matrix product $\tilde{\mathbf{a}} \mathbf{D}_e \mathbf{a}$. According to [Pustaic 1990] and [Štok 1976] we shall have:

$$\tilde{\mathbf{a}} \mathbf{D}_e \mathbf{a} = \frac{E}{1-n^2} \left[(5-4n)(\mathbf{s}_x - \mathbf{s}_y)^2 + 2(1+n)\mathbf{s}_x \mathbf{s}_y + 18(1-n)\mathbf{t}_{xy}^2 \right] \quad (21)$$

If we introduce the substitution:

$$\mathbf{w} = \frac{1}{1-n^2} \left[(5-4n)(\mathbf{s}_x - \mathbf{s}_y)^2 + 2(1+n)\mathbf{s}_x \mathbf{s}_y + 18(1-n)\mathbf{t}_{xy}^2 \right] \quad (22)$$

then the matrix product $\tilde{\mathbf{a}} \mathbf{D}_e \mathbf{a}$ assumes the form:

$$\tilde{\mathbf{a}} \mathbf{D}_e \mathbf{a} = E\mathbf{w} \quad (23)$$

The scalar quantity B will be equal [Pustaic 1990] and [Štok 1976]:

$$B = 4H' \mathbf{s}_e^2 + E\mathbf{w} = \frac{E}{1-n^2} S \quad (24)$$

where a mark S was introduced for a scalar quantity:

$$S = (1-n^2) \left(\mathbf{w} + 4\mathbf{s}_e^2 \cdot \frac{H'}{E} \right) \quad (25)$$

The corresponding matrix product $\mathbf{D}_e \mathbf{a} \tilde{\mathbf{a}} \mathbf{D}_e$ is determined in references [Pustaic 1990] and [Štok 1976] and amounts:

$$\mathbf{D}_e \mathbf{a} \tilde{\mathbf{a}} \mathbf{D}_e = \frac{E}{1-n^2} \frac{E}{1-n^2} \begin{bmatrix} \mathbf{a}^2 & \mathbf{ab} & \mathbf{ag} \\ \mathbf{ba} & \mathbf{b}^2 & \mathbf{bg} \\ \mathbf{ga} & \mathbf{gb} & \mathbf{g}^2 \end{bmatrix} \quad (26)$$

The following marks were introduced:

$$\begin{aligned} \mathbf{a} &= (2-n)\mathbf{s}_x - (1-2n)\mathbf{s}_y, \\ \mathbf{b} &= (2-n)\mathbf{s}_y - (1-2n)\mathbf{s}_x, \\ \mathbf{g} &= 3(1-n)\mathbf{t}_{xy}. \end{aligned} \quad (27)$$

On the basis of expressions (13), (24), (25), (22), (17) and (26) the final expression for the plasticity matrix \mathbf{D}_p is obtained, in the case of plane stress:

$$\mathbf{D}_p = \frac{E}{1-n^2} \cdot \frac{1}{S} \begin{bmatrix} \mathbf{a}^2 & \mathbf{ab} & \mathbf{ag} \\ \mathbf{ba} & \mathbf{b}^2 & \mathbf{bg} \\ \mathbf{ga} & \mathbf{gb} & \mathbf{g}^2 \end{bmatrix} \quad (28)$$

If the elasticity matrix \mathbf{D}_e and the plasticity matrix \mathbf{D}_p are known, then they can be condensed in one matrix, the so-called elasto-plastic matrix \mathbf{D}_{ep} . This condensation is performed on the basis of expression (12). When the elasto-plastic matrix \mathbf{D}_{ep} is known, then the elasto-plastic strain increment can be determined according to the formula (10), assuming that the stress increment vector $d\mathbf{s}$ is known. On the basis of expressions (12), (18) and (28), the matrix equation (10), in the case of plane stress, can be written in the following form:

$$\begin{Bmatrix} d\mathbf{s}_x \\ d\mathbf{s}_y \\ d\mathbf{t}_{xy} \end{Bmatrix} = \left(\frac{E}{1-n^2} \begin{bmatrix} 1 & n & 0 \\ n & 1 & 0 \\ 0 & 0 & \frac{1-n}{2} \end{bmatrix} - \frac{E}{1-n^2} \cdot \frac{1}{S} \begin{bmatrix} \mathbf{a}^2 & \mathbf{ab} & \mathbf{ag} \\ \mathbf{ba} & \mathbf{b}^2 & \mathbf{bg} \\ \mathbf{ga} & \mathbf{gb} & \mathbf{g}^2 \end{bmatrix} \right) \cdot \begin{Bmatrix} d\mathbf{e}_x \\ d\mathbf{e}_y \\ d\mathbf{g}_{xy} \end{Bmatrix} \quad (29)$$

4. Discretization of the plate quarter with circular hole on the finite elements

For an illustration of a numerical procedure, a thin rectangular plate with circular hole was modelled. The dimensions of the plate were 192 x 96 x 2 mm. The diameter of the circular hole was $d = 2r = 32$ mm. The material model can be shown schematically as elastic-perfectly plastic. The yield stress for the above mentioned material amounts $\mathbf{s}_Y = 310$ MPa. The load acts in-plane of the plate. All models were loaded on stretching.

Figure 1 shows discretization of the plate quarter with circular hole on the triangular and isoparametric quadrilateral finite elements. The triangular finite element has three nodes in the triangle points. Each node has two degrees of freedom. Therefore, each element has six degrees of freedom. Distribution of displacements by such elements follows the linear law. The stresses and strains will have the same value in all points of the finite element. The isoparametric quadrilateral finite elements have the nodes in the points of a quadrangle. Each node has two degrees of freedom. Therefore, each element has eight degrees of freedom. These elements are linear isoparametric finite elements. Discretization of the plate quarter on 286 finite elements with 171 nodes and 342 degrees of freedom is presented in figure

1a. In total, there are 28 nodes with boundary conditions. Figure 1b presents discretization of the plate quarter on the linear isoparametric quadrilateral finite elements. There are 105 finite elements altogether with 125 nodes, 252 degrees of freedom and 25 nodes with boundary conditions.

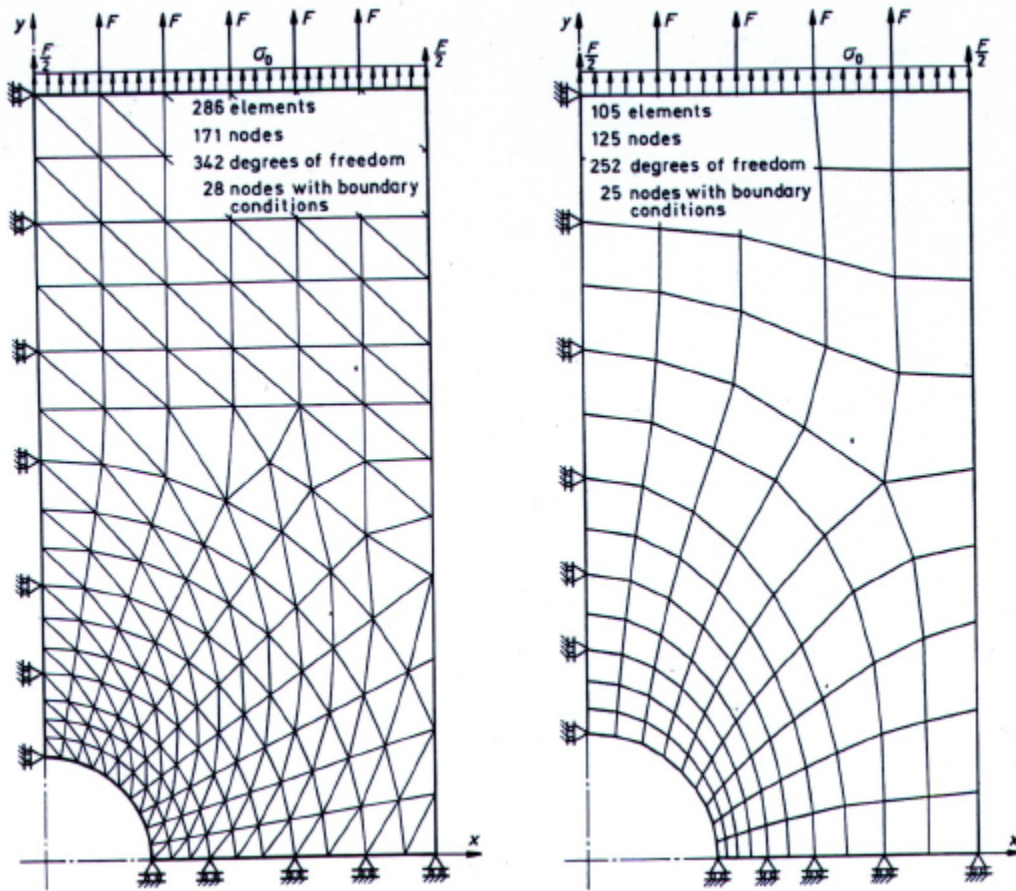


Figure 1. Discretization of the plate quarter with circular hole on the finite elements:
a) the triangular finite elements net, b) an isoparametric quadrilateral finite elements net.

5. Presentation of the results obtained by the numerical modelling

5.1 Stress fields \mathbf{s}_x and \mathbf{s}_y in the case of purely elastic and elasto-plastic state of the plate

The stress fields $\mathbf{s}_x, \mathbf{s}_y, \mathbf{t}_{xy}$ and \mathbf{s}_{equ} , as well as the displacement fields u and v were determined by the numerical analysis in all finite elements. The solutions in the elastic and elasto-plastic range will be presented further in the text. The elastic solution was obtained by external load level $\mathbf{s}_o = \frac{3}{4} \mathbf{s}_Y$.

On the other hand, the elasto-plastic solutions were obtained for three different values of external load, i. e. $0.6 \mathbf{s}_o$, $0.8 \mathbf{s}_o$ and $1.0 \mathbf{s}_o$. Figure 2a shows the stress distributions \mathbf{s}_x and \mathbf{s}_y in the characteristic sections of the plate in elastic and elasto-plastic range, obtained by the triangular finite elements with linear variation of displacements within the element. From figure 2a it is clear that the stress concentration occurs near to the boundary of discontinuity. This phenomenon is more stressed in the elastic range than in the elasto-plastic one. The stress \mathbf{s}_y is almost constant in the characteristic section of the plate, along axis x , in the plastic range. It is a consequence of elastic-perfectly plastic

material model. In figure 2b the stress field S_y , obtained by the linear isoparametric quadrilateral finite elements is shown, both in the elastic and plastic range.

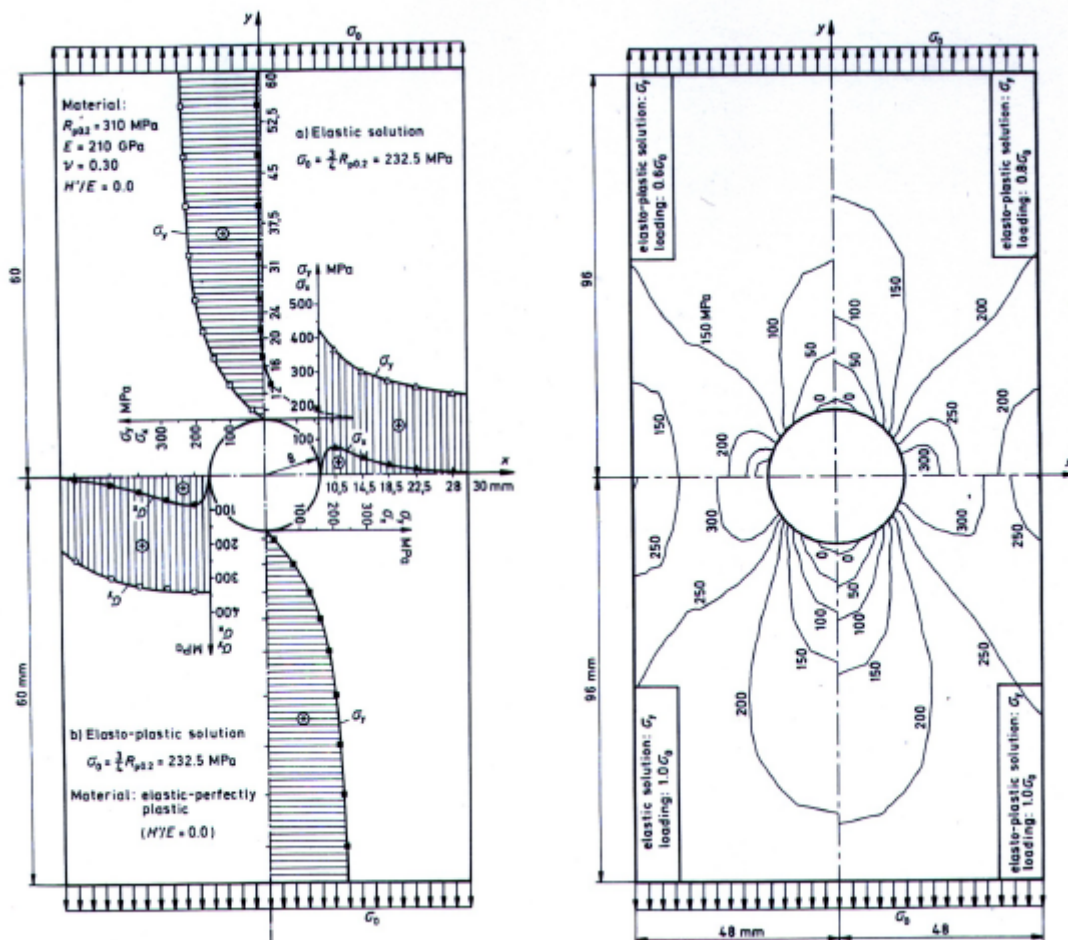


Figure 2. a) Normal stresses S_x and S_y distributions in some characteristic sections of the plate, b) stress field S_y in elastic and plastic range; the result obtained with the linear isoparametric quadrilateral finite elements

5.2 Presentation of the plastic zones spreading with increasing intensity of external loads

By the numerical analysis of a structure in the non-linear range, the external loads are applied in increments. It gives a possibility of locating the occurrence of first plastic strains on the boundaries of circular discontinuity, or a possibility of keeping track of plastic regions spreading around geometrical discontinuity as external loads are increased. The figure 3.a) shows the shape and the magnitude of the plastic zones around the circular hole in the plate, and spreading of these zones with the increasing intensity of external loads. These solutions were obtained with the triangular constant strains finite elements, while those from figure 3.b) are the result of a problem analysis with the linear isoparametric quadrilateral finite elements. Comparing the shape and the magnitude of the plastic zones in figures 3.a and 3.b a good agreement of the obtained results for the same level of the external loads can be confirmed.

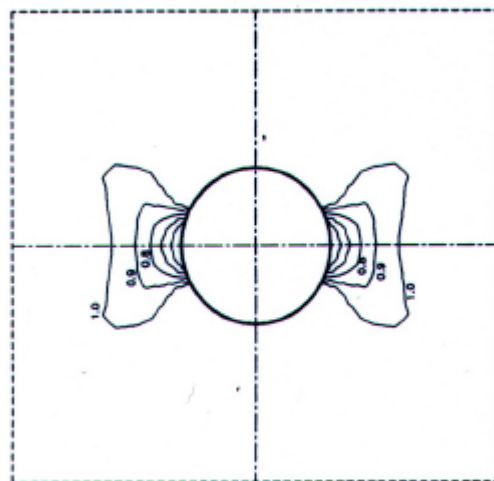
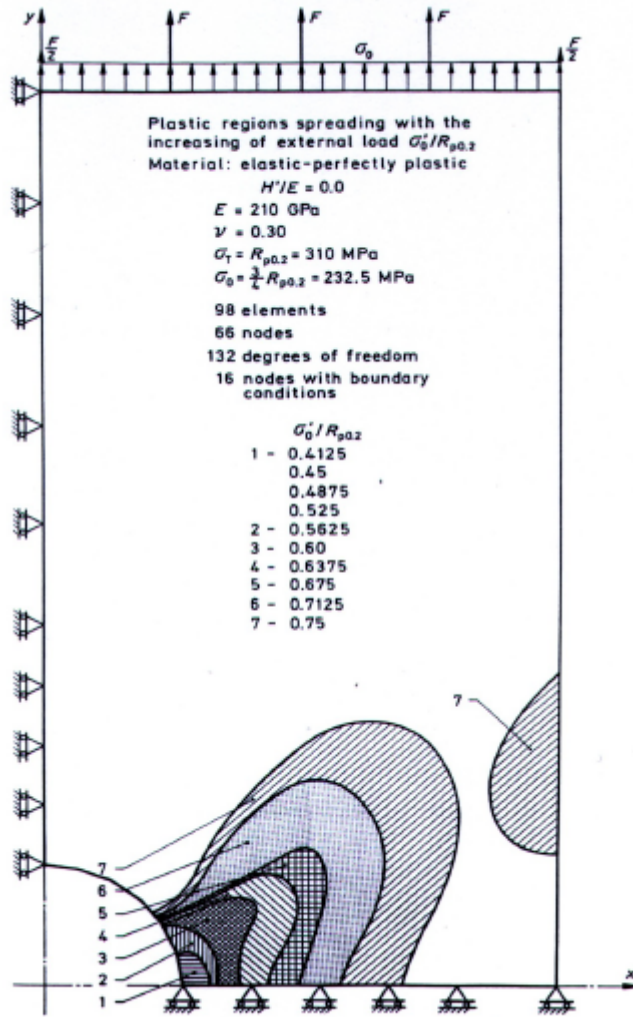


Figure 3. Plastic regions spreading around geometrical discontinuity with the increasing intensity of the external load, a) the solution obtained with the triangular constant strains finite elements, b) the result of a problem analysis with the linear isoparametric quadrilateral finite elements.

6. Conclusion

A thin rectangular plate with the circular hole is modeled in order to illustrate a numerical procedure for determination of the stresses, strains and displacements. Using the finite elements nets from figure 1, the stress fields \mathbf{s}_x , \mathbf{s}_y , \mathbf{t}_{xy} and \mathbf{s}_{equ} , as well as the displacement field u and v , were computed in the elastic and elastic-plastic range. Some of the solutions are presented in figure 2. In the plastic range the stress distribution \mathbf{s}_y in the section across x-axis is completely different from that in the elastic range. The stress \mathbf{s}_y is almost constant through the whole mentioned section. It is a consequence of an elastic-perfectly plastic material model of the plate. By numerical analysis of the structure in a non-linear range the external loads are applied in increments. It gives a possibility of keeping track of the plastic zones spreading around the geometrical discontinuity as external loads are increased. Figure 3.a shows the shape and the magnitude of the plastic zones around the circular hole in the plate which are obtained with the triangular finite elements. The solution from the figure 3.b is the result of a problem analysis with the linear isoparametric quadrilateral finite elements. Comparing the shape and the magnitude of the plastic zones in figure 3 a good agreement of the obtained results for the same level of external loads can be confirmed. It is true both for the case when the plate is modeled with the triangular finite elements as well as for the case with isoparametric quadrilateral ones.

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