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CURVE SHAPE MODIFICATION AND FAIRNESS EVALUATION FOR COMPUTER AIDED AESTHETIC DESIGN

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ABSTRACT
For the purpose of evaluation, a NURBS curve is used, because it is commonly used in the areas of CAD・CAM and Computer Graphics. A curve with a monotone radius of curvature distribution is considered as a fair curve in the area of Computer Aided Aesthetic Design (CAAD). But no official standards have been established. Therefore, a criterion for a fair curve is proposed. The designed curve is assumed fair if the variation of radius of curvature is followed by either linear, quadratic, or cubic algebraic functions for the same number of control points and knot sequences of the knot vectors. Using these three algebraic functions as specified radius of curvature distributions, a curve shape modification algorithm based on these specified radius of curvatures is applied. The sum of the squared difference between the radius of curvature of the curve and the specified radius of curvature of an entire curve is linearized by Taylor's theorem, then minimized. If the similarity of the original curve is very close to one of the three shape modified curves, it can be judged that the original curve is designed according to the designer's intention, which is one of the three radius of curvature distributions. This measured similarity expresses fairness to the fair curve.

Keywords: Curve shape modification, fair curve, radius of curvature specification, correlation matching, fairness evaluation

1 INTRODUCTION

In Computer Aided Aesthetic Design, designers evaluate the quality of a designed curve by looking at its curvature or radius of curvature plots. If the quality of a designed curve does not meet the designer's demands, they usually modify the control points of the curve interactively. If the variation of the radius of curvature of the curve is monotone, this curve is considered to be a fair curve [1]. But the definition of a fair curve is ambiguous and no official standards are given. Therefore, in this paper we have tried to establish criterion for a fair curve. For a curve fairness evaluation measurement, radius of curvature distribution is used as an alternative characteristic of a curve. Evaluation of whether the designed curve is fair or not is accomplished by comparing of the designed curve to a curve whose radius of curvature is monotone.

A NURBS curve, which is commonly used in the field of CAD・CAM and Computer Graphics, is used as an expression of a freeform curve. Three types of NURBS curves were considered. They are the quadratic, cubic, and quintic NURBS curves. A quadratic NURBS curve is expressed as a quadratic curve by using its weights. In this study, a quadratic curve is not used to express the shape of a fair curve. Therefore, the weights of a quadratic NURBS curve are not used. A cubic NURBS curve is also widely used, but in this study, radius of curvature ranging over multi segments of a NURBS curve are modified based on the specified radius of curvature. A smooth radius of curvature continuity is needed. Therefore, a quintic NURBS curve is used in this study. It is assumed that no inflection points exist on a quintic NURBS curve.

There are many related works for generation of a fair curve dealing with knots. These are knot insertion [2, 3], knot removal algorithm for B-spline curves [4], knot removal algorithm for NURBS curves [5], and fair curve generation by knot value and weight modification [6].

There are many related works for generation of fair curvature distribution. A fair curvature distribution algorithm by modifying knot spacing [7, 8], and by removing and reinserting knots [9-13] have been published.
Fair curve generation algorithms related to curvature by modifying the control points have been published. These make monotone curvature [14], use a clothoidal curve for specifying the curvature [15], and automate a curve fairing algorithm for B-spline curves [16, 17]. Fair curve generation algorithms related to energy function have been published. These are smoothing of cubic parametric splines by energy function [18], finding the unfair portion of a curve using energy function [19], and introducing a low-pass filter to energy function [20]. Fair curve generation algorithms related to curvature by specifying curvature distribution have also been published [21].
There are many related works for evaluating similarities of polygons in two dimensional space, especially in the area of image processing. Methods for evaluating similarities, which are based on the distances of corresponding points on polygonal curves, have been reported [22-25]. If the distances are close, it will be determined that the two polygonal curves are similar. Methods using Fourier descriptors for evaluating similar polygons have been developed and implemented [26, 27]. One is to retrieve the image files using Fourier descriptors. The other is to classify the characters expressed by polygonal curves.
Section 2 of this paper describes a quintic NURBS curve, the first derivative of a quintic NURBS curve, curvature vector, curvature, and radius of curvature. In section 3, NURBS curve shape modification based on the specified radius of curvature is described. Section 4 describes the correlation matching to evaluate the similarity of the NURBS curves. Section 5 describes fair curve expression and fairness evaluation giving examples. A criterion for a fair curve is proposed.

2 NURBS CURVE EXPRESSION

A quintic NURBS curve is used in this study. The objective of freeform curve design is to design the framework of surface patches. Surface patches are defined as tensor products, which are bi-variate and normally defined by u and v. In other words, one knot sequence in u direction, and another knot sequence in v direction are defined despite the complexity of the surface patches. Therefore, knot spacing is fixed in this study.
A quintic NURBS curve consists of $n-5$ segments $(n \geq 6)$ is composed of n control points such as $\boldsymbol{q_0}, \boldsymbol{q_1}, \cdots, \boldsymbol{q_{n-1}}$ and n weights such as $\omega_0, \omega_1, \cdots, \omega_{n-1}$ as in Eq.(1).

$$\boldsymbol{R}(t) = \frac{\sum_{i=0}^{n-1} N_{i,6}(t) \cdot \omega_i \cdot \boldsymbol{q}_i}{\sum_{i=0}^{n-1} N_{i,6}(t) \cdot \omega_i}, \tag{1}$$

where $N_{i,6}(t)$ $(i = 0,1,\cdots,n-1)$ are NURBS basis functions.
These functions are recursively defined by knot sequence $t_0, t_1, \cdots, t_{n+5}$ as in Eq.(2).

$$\left.\begin{aligned} N_{i,1}(t) &= \begin{cases} 1 & (t_i \leq t < t_{i+1}) \\ 0 & \text{otherwise} \end{cases} \\ N_{i,M}(t) &= \frac{t - t_i}{t_{i+M-1} - t_i} N_{i,M-1}(t) + \frac{t_{i+M} - t}{t_{i+M} - t_{i+1}} N_{i+1,M-1}(t) \end{aligned}\right\}, \tag{2}$$

where $i = 0,1,\cdots,n-1$ and $M = 2,3,\cdots,6$.
The basis functions are defined by the de Boor-Cox [28] recursion formulas. If the knot vector contains a sufficient number of repeated knot values, then a division of the form $N_{i,M-1}(t)/(t_{i+M-1} - t_i) = 0/0$ (for some i) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$ [29].
Curvature vector is expressed by Eq.(3).

$$\boldsymbol{\kappa}(t) = \frac{\left(\dot{\boldsymbol{R}}(t) \times \ddot{\boldsymbol{R}}(t)\right) \times \dot{\boldsymbol{R}}(t)}{\left(\dot{\boldsymbol{R}}(t)\right)^4}, \tag{3}$$

where $\dot{\boldsymbol{R}}(t)$ is the first derivative of a NURBS curve, and $\ddot{\boldsymbol{R}}(t)$ is the second derivative of a NURBS curve. Curvature is the magnitude of the curvature vector, therefore curvature is expressed as in Eq.(4).

$$\kappa(t) = \left|\boldsymbol{\kappa}(t)\right| \tag{4}$$

By definition, the curvature of a plane curve is non-negative. However, in many cases it is useful to ascribe a sign to the curve [30]. The choosing of the sign is commonly connected with the tangent rotation (in moving along the curve in the direction of the increasing parameter): The curvature of the curve is positive when its tangent rotates counter-clockwise, the curvature of the curve is negative when its tangent rotates clockwise.

Radius of curvature is the reciprocal number of curvature, therefore, radius of curvature is expressed as in Eq.(5).

$$\rho(t) = \frac{1}{\boldsymbol{\kappa}(t)} \tag{5}$$

3 NURBS CURVE SHAPE MODIFICATION BASED ON THE SPECIFIED RADIUS OF CURVATURE

A method to modify a NURBS curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described in this section. The difference between the NURBS curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method.

If the radius of curvature to the perimeter is linear, curvature distribution will be parabolic. On the contrary, if the curvature to the perimeter is linear, the radius of curvature distribution will be parabolic. Radius of curvature is suitable for our use because it corresponds to our visual recognition of the shape of the curve. In a case where the curve shape is very close to a straight line, the radius of curvature becomes infinity. Also, at the point of inflexion, curvature value becomes zero. Therefore, radius of curvature value becomes infinite. For these reasons, the radius of curvature value is converted to curvature value for computation.

The concept of radius of curvature specification and a NURBS curve shape modification based on the specified radius of curvature is shown in Figure 1. A NURBS curve and its radius of curvature plots are shown in Figure 1(a). The modification of the shape of the NURBS curve shown in Figure 1(a) to that shown in Figure 1(b) is examined. Radius of curvature plots shown in Figure 1(a) are drawn perpendicular to the curve using straight lines. The length of the line is proportional to the radius of curvature at the spot on the curve. However, the straight lines are not parallel to each other and the beginning points of the individual straight lines are different. Therefore, the curve with a radius of curvature display is suitable to examine the variation of radius of curvature as a whole. However, it is not suitable to examine the length of the straight line and variation of radius of curvature visually.

Therefore, considering the parameter of the NURBS curve is different from the perimeter of the curve, the perimeter of the NURBS curve as a straight line is set to the horizontal axis, and radius of curvature is set to the vertical axis as shown in Figure 1(c). Then, the radius of curvature distribution to the perimeter is drawn. After this, specified radius of curvature is superimposed on the current radius of curvature distribution.

As an example, the linear algebraic function as a specified radius of curvature specification is shown in Figure 1(c). Coefficients of this linear function are calculated by introducing the least-squares method using the current radius of curvature distribution.

The i th of radius of curvature distribution of a perimetrically represented NURBS curve is denoted ρ_i, the specified radius of curvature at the same spot is denoted $\hat{\rho}_i$, the difference δ_i is shown by Eq.(6) and is illustrated in Figure 1(c).

$$\delta_i = \rho_i(q_1^x, \cdots, q_{n-2}^x, q_1^y, \cdots, q_{n-2}^y) - \hat{\rho}_i \tag{6}$$

where $i = 0,1,2,\cdots,m-1$, m is number of the specified radius of curvature, and n is the number of NURBS curve segments plus 5 which is the degree of the curve.

$S(q_1^x, \cdots, q_{n-2}^x, q_1^y, \cdots, q_{n-2}^y)$ which is the sum of the squared differences for all specified radius of curvatures in Eq.(7) is minimized by introducing the least-squares method. The radius of curvature expression is non-linear. Therefore, by Taylor's theorem, Eq.(7) is linearized as in Eq.(8).

$$S(q_1^x, \cdots, q_{n-2}^x, q_1^y, \cdots, q_{n-2}^y) = \sum_{i=0}^{m-1}\left[\rho_i(q_1^x, \cdots, q_{n-2}^x, q_1^y, \cdots, q_{n-2}^y) - \hat{\rho}_i\right]^2 \tag{7}$$

$$S(q_1^x + \Delta q_1^x, \cdots, q_{n-2}^x + \Delta q_{n-2}^x, q_1^y + \Delta q_1^y, \cdots, q_{n-2}^y + \Delta q_{n-2}^y)$$
$$= \sum_{i=0}^{m-1}\left[\rho_i(q_1^x, \cdots, q_{n-2}^x, q_1^y, \cdots, q_{n-2}^y) + \frac{\partial \rho_i}{\partial q_1^x}\Delta q_1^x + \cdots + \frac{\partial \rho_i}{\partial q_{n-2}^x}\Delta q_{n-2}^x + \frac{\partial \rho_i}{\partial q_1^y}\Delta q_1^y + \cdots + \frac{\partial \rho_i}{\partial q_{n-2}^y}\Delta q_{n-2}^y - \hat{\rho}_i\right]^2 \tag{8}$$

To minimize Eq.(8) is achieved by equating to zero all the partial derivatives of $S(q_1^x + \Delta q_1^x, \cdots, q_{n-2}^x + \Delta q_{n-2}^x, q_1^y + \Delta q_1^y, \cdots, q_{n-2}^y + \Delta q_{n-2}^y)$ with respect to Δq_r^x and Δq_r^y $(r = 1, 2, \cdots, n-2)$ as in Eq.(9).

$$\left.\begin{array}{l} \dfrac{\partial S}{\partial \Delta q_r^x} = 0 \quad (r = 1, 2, \cdots, n-2) \\ \dfrac{\partial S}{\partial \Delta q_r^y} = 0 \quad (r = 1, 2, \cdots, n-2) \end{array}\right\} \tag{9}$$

Using these simultaneous linear equations, Δq_r^x and Δq_r^y $(r = 1, 2, \cdots, n-2)$ are calculated. Then, q_r^x, q_r^y are determined.

A reverse computation technique is applied to solve this problem. This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem [31]. For computation, ρ_i and $\hat{\rho}_i$ are calculated based on the perimeter. Then, the perimeter used is converted to the parameter to calculate the position of the control points of the NURBS curve. Next, a NURBS curve is generated. The total length of the curve which is the perimeter is calculated and rescaled as 1. Repeating these operations, positions of the control points of the NURBS curve are determined while δ_i $(i = 0, 1, \cdots, m-1)$ are minimized for the entire perimeter.

Using the above mentioned method of a linear algebraic function to specify the radius of curvature shown in Figure 1(c), radius of curvature distribution is changed to the one shown in Figure 1(d), while modifying the shape of the curve. The dotted line shown in Figure 1(d) is a linear algebraic function specifying the radius of curvature distribution shown in Figure 1(c). It is visually recognized that the radius of curvature distribution of the shape modified curve shown in Figure 1(d) matches the specified radius of curvature.

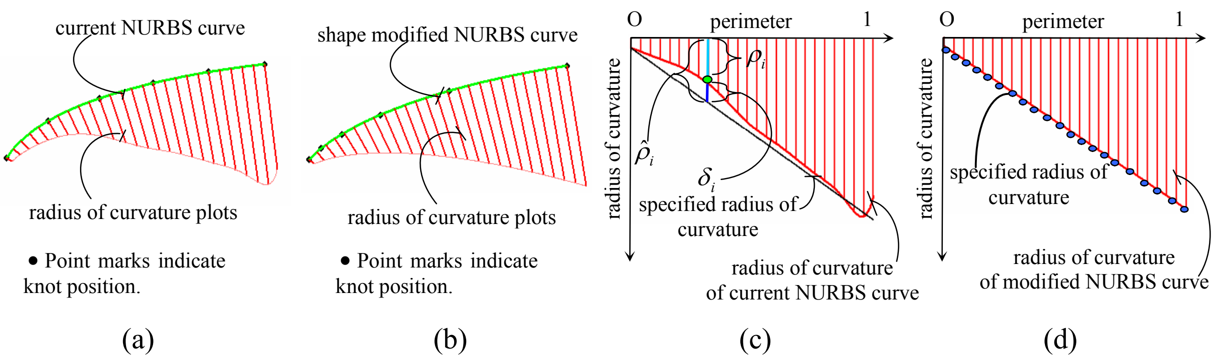


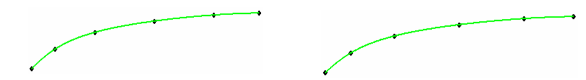
Figure 1. Concept of radius of curvature specification and NURBS curve shape modification based on the specified radius of curvature

(a) : current NURBS curve and its radius of curvature plots
(b) : shape modified NURBS curve and its radius of curvature plots
(c) : difference between current radius of curvature and specified radius of curvature
(d) : radius of curvature of shape modified NURBS curve
and specified radius of curvature (same as in (c))

4 CORRELATION MATCHING FOR SIMILARITY EVALUATION

In this section, correlation matching for similarity evaluation is described. Radius of curvature distribution is used as an alternative characteristic of the shape of the curve to evaluate a designed curve. Discrete values, which are radius of curvature to the perimeter of the reference, and the matching curve are considered as the components of two multi dimensional vectors. Similarity is evaluated by the dot product of two vectors. Two NURBS curves are shown in Figure 2(a) and 2(b). These curves can hardly be distinguished apart by just looking at their graphs. But if the radius of curvature plots are drawn for both, the difference between the two curves is recognized immediately as shown in Figure 3(a) and 3(b).

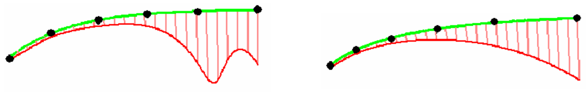
Radius of curvature is plotted using straight lines drawn outward from and perpendicular to the curve, with the line length proportional to the amount of radius of curvature at that spot. Curve shape is judged by looking at the lines coming out from the curve and seeing how their lengths change along the path, not along the parameter. Therefore, radius of curvature to the perimeter is drawn to evaluate the similarity of the curve shape as shown in Figure 4(a) and 4(b).



(a) curve A (b) curve B

• Point marks indicate knot position.

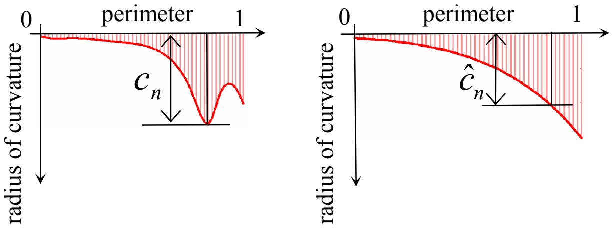
Figure 2. Two NURBS curves



(a) curve A (b) curve B

• Point marks indicate knot position.

Figure 3. Two NURBS curves with radius of curvature plots



(a) radius of curvature distribution for curve A (b) radius of curvature distribution for curve B

Figure 4. Radius of curvature distribution of two NURBS curves

To adjust the various lengths of the curve perimeters, the total length of the perimeters and radius of curvature are rescaled as 1. A perimeter must be calculated according to the knot sequence of the knot vector.

Discrete values $c_n (n = 1, 2, 3, \cdots m)$ shown in Figure 4(a) which are radius of curvature to the perimeter are considered as the components of m dimensional vector for curve A, denoting $\mathbf{a}$. In the same manner, discrete values $\hat{c}_n$ shown in Figure 4(b) are considered as the components of m dimensional vector for curve B, denoting $\mathbf{b}$.

Similarity between curve A and curve B is evaluated by Eq.(10).

$$S = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \quad (10)$$

Because curve perimeter and radius of curvature are rescaled as 1, and perimeter is calculated according to the knot sequence of the knot vector, the similarity is evaluated independent of location, orientation such as rotation and reflection, and the size of the curves. The evaluated similarity between curve A and curve B shown in Figure 4 is 0.996792.

5 FAIR CURVE EXPRESSION AND EVALUATION OF FAIRNESS

In this section, fair curve expression and evaluation of fairness are described. As a measure of curve fairness evaluation, radius of curvature distribution is used as an alternative characteristic of a curve.

A curve with a monotone radius of curvature distribution is considered as a fair curve in the area of Computer Aided Aesthetic Design [32]. But no official standards are given. Therefore, criterion for a fair curve is tentatively proposed.

The shape of a NURBS curve is defined by the number, the location of its control points, and the knot sequence of the knot vector. The designed curve is assumed fair if the variation of radius of curvature is monotone for the same number of control points and knot sequence of the knot vector.

First, radius of curvature distribution of the designed curve is calculated to examine the fairness of the curve. Algebraic functions such as linear, quadratic, and cubic are applied to the radius of curvature distribution of the designed curve by introducing the least-squares method. Then applying the curve shape modification algorithm based on the specified radius of curvature distribution, the radius of curvature distribution is modified according to these three algebraic functions respectively.

These radius of curvature distributions given by these three algebraic functions are considered monotone, because the independent variable of these algebraic functions is monotone to the corresponding dependent variable of these functions. Therefore, curves designed in this manner are considered fair. These judgements are performed by evaluating the similarity technique described in the previous section. This similarity is evaluated by using the radius of curvature distribution according to three algebraic functions used as references and the radius of curvature distribution of the designed curve as a match.

It is considered that the highest similarity reveals that this curve is designed to have this radius of curvature distribution, and the similarity measured is considered as the fairness of the designed curve.

As an example of a fair curve generation and fairness evaluation, a NURBS curve and its radius of curvature distribution to the perimeter are shown in Figure 5.

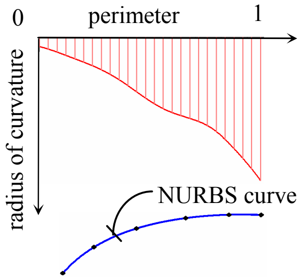


Figure 5. Designed curve and its radius of curvature distribution

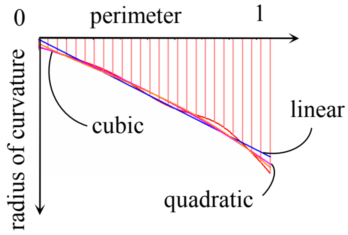


Figure 6. Radius of curvature distribution and a given algebraic function such as linear, quadratic, and cubic to specify radius of curvature

Algebraic functions mentioned above are applied to the radius of curvature distribution shown in Figure 5 by using the least-squares method. Then, the coefficients of these three functions are calculated. These three functions are determined as the specified radius of curvature. These are shown in Figure 6 together with the radius of curvature distribution shown in Figure 5. Applying the curve shape modification algorithm based on these three algebraic functions to the designed curve, the shape of the curve is modified. Afterwards, setting these three radius of curvature distributions as references and the radius of curvature distribution of the designed curve as a match, three similarities are evaluated. The similarities evaluated are summarized in Table 1. In Table 1, similarity expresses the fairness of the curve. In addition to Table 1, the similarity expressed in degree is summarized in Table 2. From Table 1 and Table 2, the designed curve whose radius of curvature is shown in Figure 5, is judged to be designed so that the radius of curvature distribution will be cubic. Then, the fairness of this curve is evaluated as 0.996082.

Table 1. Fairness of the designed curve evaluated in cosine

	Linear	Quadratic	Cubic
Radius of curvature shown in Figure 5	0.989403	0.994786	0.996082

Table 2. Fairness of the designed curve evaluated in degree

	Linear	Quadratic	Cubic
Radius of curvature shown in Figure 5	3.86192	2.70623	2.49432

Giving eight sample curves, fairness is examined. Eight designed curves and their radius of curvature distributions are shown in Figure 7 (a), (b), (c), (d), (e), (f), (g), (h), and are labeled curve A, B, C, D, E, F, G, H respectively.

Applying the three algebraic functions to the radius of curvature distributions of these eight curves respectively, the radius of curvature distributions corresponding to these three algebraic functions are generated. If the radius of curvature is negative, it is considered that this algebraic function is not applicable. The original radius of curvature distributions of eight designed curves and that of modified curves based on the specified algebraic functions are shown in Figure 8.

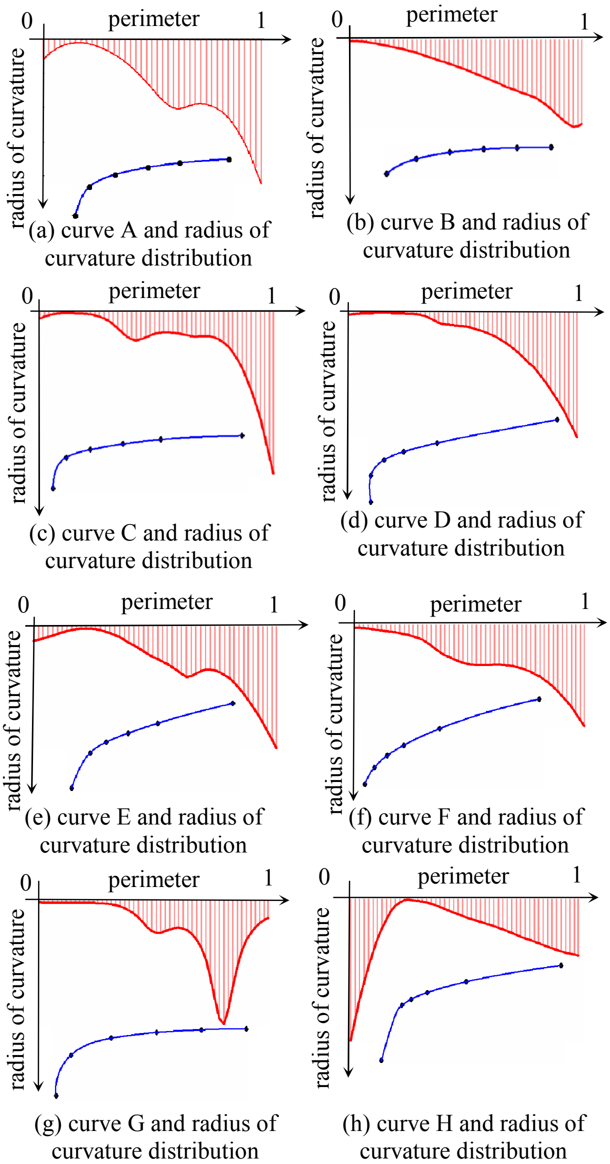
The similarity is evaluated by using the radius of curvature distribution according to these three algebraic functions as references and the radius of curvature distribution of the eight designed curves shown in Figure 7 as matches.

The radius of curvature distributions given by these three algebraic functions are considered monotone. Therefore, it is also considered that the curves with monotone radius of curvature distribution are fair. So, the similarity evaluated is considered fairness. The fairness of eight curves shown in Figure 7 is summarized in Table 3.

The similarity measured using Eq.(10) is expressed in cosine. To distinguish the small differences, the similarity is expressed in degree and summarized in Table 4.

From Table 4, it is recognized that curve B is designed so that its radius of curvature distribution will follow quadratic function as shown by the yellow hatching. Notice that the fairness of curve B is evaluated as 0.999420. It is also visually recognized that curve B's radius of curvature distribution fits to that of quadratic as shown in Figure 8(b). In addition to this, it is recognized that curve F is designed so that its radius of curvature distribution will follow quadratic function as shown by the green hatching. The fairness of curve F is evaluated as 0.990694. But it is recognized that curve C, D, G and H are designed according to the designer's unique intention as shown by the red hatching.

As mentioned above, fairness of the designed curve is proposed by a similarity evaluation technique using the radius of curvature distribution.



・point marks on the indicate knot position

Figure 7. Designed curves and their radius of curvature distributions

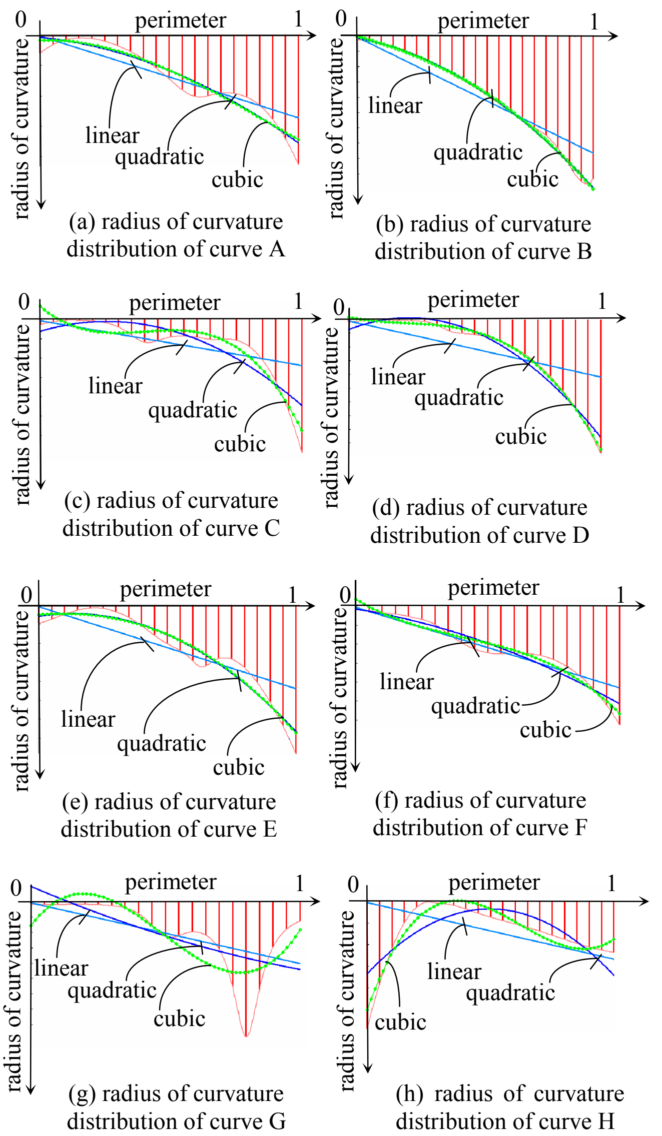


Figure 8. Radius of curvature distribution of individual curve and the radius of curvature corresponding to linear, quadratic, and cubic functions

Table 3. Fairness of the designed curves evaluated in cosine

	A	B	C	D	E	F	G	H
Linear	0.952451	0.966819	0.781774	0.816568	0.923317	0.969153	0.804327	0.729839
Quadratic	0.985327	0.999420	0.938154	-	0.987085	0.990694	-	0.917646
Cubic	0.986855	0.999398	-	-	0.989634	-	-	-

Table 4. Fairness of the designed curves evaluated in degree

	A	B	C	D	E	F	G	H
Linear	17.739571	14.800924	38.576668	35.257306	22.584137	14.268186	36.454733	43.127084
Quadratic	9.827210	1.951571	20.256155	-	9.218513	7.822708	-	23.415688
Cubic	9.300208	1.988754	-	-	8.256947	-	-	-

6 CONCLUSIONS

A quintic NURBS curve, the first derivative of a quintic NURBS curve, curvature vector, curvature, and radius of curvature are described.

A method to modify NURBS curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described. The difference between the NURBS curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method. A reverse computational technique is applied to solve this problem. This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem.

Correlation matching for similarity evaluation is described. The values of radius of curvature to the perimeter are considered as the components of a multi dimensional vector for the curve. Similarity between two curves is expressed by normalizing the dot product of two vectors. Curve shape similarity evaluation is tried using an example.

A curve with a monotone radius of curvature distribution is considered as a fair curve in the area of Computer Aided Aesthetic Design. A criterion for a fair curve is proposed. Evaluation whether the designed curve is fair or not is accomplished by a comparison of the designed curve to a curve whose radius of curvature is monotone. The radius of curvature is specified by linear, quadratic, and cubic function using the least-squares method.

The fairness of a curve is evaluated by using the similarity of the radius of curvature distribution.

If the similarity of the original curve is very close to one of the three shape modified curves, it can be judged that the original curve is designed according to the designer's intention, which is one of the three radius of curvature distributions. This measured similarity expresses the fairness of the fair curve.

In the future, we are planning to establish a definition of a fair curve using further curve data that will be gathered.

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